The Mathematics of DÉjà Vu

2026-27

Mathematics Club

Centre for Innovation
IIT Madras

Instructions

General

- Provide descriptive answers for all questions. Explain your thought process well.
- You are free to use the internet and any other resources for solving the application. Please cite whatever reference(s) you use.
- The goal of the application is for you to learn as much as possible. Resources have been provided in the end for all sections. All the best and have fun!

Assignment Format

- Your solutions can be written manually, digitally, documents or in LaTeX. Submit all your scanned copies with the following format `<FullNameSS.pdf>`. For example, if my name is Arjun Arunachalam, I would submit `ArjunArunachalamSS.pdf`.

Submission

- The final PDF is to be submitted in this Google form- https://forms.gle/dUe5ZNBh6W1Vnzep9
- The deadline for submission is **20th July 2026, 23:59**.
- You do not have to attempt the bonus question. If you do attempt it, there is no need to submit the code, it is just for you to have fun.

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§1 Recurrence Practice Problems

1. Find the n^{th} term of the sequence $\{a_n\}$ which is defined by

$$a_1 = 0, \quad a_n = \left(1 - \frac{1}{n}\right)^3 a_{n+1} + \frac{n-1}{n^2} \quad (n = 1, 2, 3, \dots).$$

2. Find the n^{th} term of the sequence $\{a_n\}$ such that

$$a_1 = 1, \quad a_2 = 3, \quad a_{n+1} - 3a_n + 2a_{n-1} = 2^n \quad (n \geq 2).$$

   P.S. *Apply the Collision Rule!*

3. Find the n^{th} term of the sequence $\{a_n\}$ such that

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{3}, \quad a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1} + 2a_n a_{n+1}}.$$

   P.S. *Think of a unique substitution before you start!*

§2 A few thoughts on the characteristic equation for solving recurrences

§2.1 Splitting the middle term

Say we are given the following recurrence relation:

$$a_n = 5a_{n-1} - 6a_{n-2}$$

How could we go about analyzing this? Imagine a situation where it is just $a_n = 5a_{n-1}$ (without the third term), life would be a lot simpler right? Why is it simpler? Because this follows a simple G.P!

Let's see if we can achieve this with our recurrence relation too. Here is an interesting idea, let's split the middle term as follows -

$$\mathbf{a_n} - \mathbf{2a_{n-1}} = 3a_{n-1} - 6a_{n-2} = \mathbf{3}(\mathbf{a_{n-1}} - \mathbf{2a_{n-2}})$$

If we define $b_n = a_n - 2a_{n-1}$, this relation becomes: $b_n = 3b_{n-1}$ which is a simple G.P! So depending on what the initial conditions are, we will get $b_n = \lambda \cdot 3^n$. But we are in pursuit of a_n, not b_n, so let's substitute the formula for b_n back in here:

$$a_n - 2a_{n-1} = \lambda \cdot 3^n$$

This is now a new problem to solve, how could we solve this? It really feels like this should telescope, my soul really wants this to telescope, but that darn **2** is irritating me! What can I do?

I can do the following, multiply each successive equation by higher powers of 2 to force a telescope:

$$2^0(a_n - 2a_{n-1} = \lambda \cdot 3^n)$$
$$2^1(a_{n-1} - 2a_{n-2} = \lambda \cdot 3^{n-1})$$
$$2^2(a_{n-2} - 2a_{n-3} = \lambda \cdot 3^{n-2})$$

...and so on till...

$$2^{n-2}(a_2 - 2a_1 = \lambda \cdot 3^2)$$

This nicely telescopes and we get:

$$a_n - 2^{n-1}a_1 = \lambda \cdot \sum_{i=0}^{n-2} 2^i \cdot 3^{n-i}$$

The RHS is another G.P. (G.P.s galore):

$$\sum_{i=0}^{n-2} 2^i \cdot 3^{n-i} = 3^n \sum_{i=0}^{n-2} \left(\frac{2}{3}\right)^i = 3^n \cdot \frac{1-(2/3)^{n-1}}{1/3} = 9 \cdot (3^{n-1} - 2^{n-1})$$

So, (almost) finally,

$$a_n = 2^{n-1}a_1 + 9\lambda \cdot (3^{n-1} - 2^{n-1})$$

Combining some like terms, we finally get:

$$a_n = \left(\frac{a_1 + 9\lambda}{2}\right) \cdot 2^n + \left(\frac{9\lambda}{3}\right) 3^n$$

Or, put in a simpler way:

$$a_n = (c_1) \cdot 2^n + (c_2) \cdot 3^n, \text{ where the coefficients are constants determined by the initial conditions}$$

§2.2 A Quick Review: Where else have you split the middle term?

Our entire strategy relied on the idea of splitting the middle term. Where have we split the middle term before? While solving quadratic equations! And the two numbers we get after splitting the middle term are the roots of the quadratic.

Look at the above derivation, I've highlighted 2 and 3 to emphasize the point that the two roots we got by splitting that middle carry down all the way till the end, to the final form.

This is the idea behind the "guess" discussed in the session today! Notice how the process of splitting the middle term here is very similar to the process of finding the roots of the equation $x^2 - 5x + 6 = (x-2)(x-3) = 0$.

§2.3 If the roots are equal?

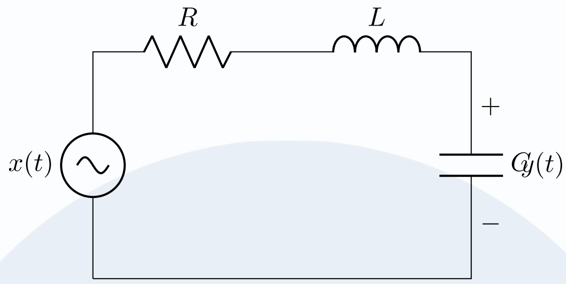
Can you go through the same derivation as above for the recurrence given below and see what changes? Why do we multiply by an n?

$$a_n = 6a_{n-1} - 9a_{n-2}$$

§3 Applications: RLC Circuits, the Dirac Delta, and Chaos

§3.1 An RLC Circuit and the Laplace Transform

Let us take the following series LCR circuit into consideration.



The circuit consists of a resistor of resistance R, an inductor of inductance L, and a capacitor of capacitance C connected in series. The input voltage is denoted by $x(t)$, while the output voltage across the capacitor is denoted by $y(t)$.

The voltage–current relationships for the individual circuit elements are given by

$$v_R(t) = Ri(t), \qquad v_L(t) = L\frac{di(t)}{dt}, \qquad i(t) = C\frac{dy(t)}{dt},$$

where $i(t)$ denotes the current flowing through the circuit.

Using Kirchhoff's Voltage Law,

$$v_R(t) + v_L(t) + y(t) = x(t),$$

answer the following questions.

1. Show that the capacitor voltage $y(t)$ satisfies the second-order linear differential equation

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t).$$

2. Explain why two initial conditions are required to uniquely determine the solution of the above differential equation.

3. Suppose the capacitor voltage and its derivative satisfy the initial conditions

$$y(0^-) = a, \qquad y'(0^-) = b,$$

where $a, b \in \mathbb{R}$. Let

$$Y(s) = \mathcal{L}_u\{y(t)\}, \qquad X(s) = \mathcal{L}_u\{x(t)\},$$

denote the unilateral Laplace transforms of the output and input voltages, respectively.

(a) Using the unilateral Laplace Transform, derive a general expression for $Y(s)$ in terms of $X(s)$, the circuit parameters L, R, C, and the initial conditions a and b.

(b) Now consider the circuit with

$$L = 1\text{ H}, \qquad R = 2\Omega, \qquad C = 1\text{ F},$$

and the initial conditions

$$y(0^-) = 1, \qquad y'(0^-) = 0.$$

At time $t = 0$, a DC voltage source of magnitude 5 V is connected to the circuit so that

$$x(t) = 5u(t),$$

where $u(t)$ denotes the unit step function.

Using the result obtained in part (a), determine the capacitor voltage $y(t)$ for $t > 0$ by first obtaining $Y(s)$ and then evaluating its inverse Laplace Transform.

§3.2 The Dirac Delta Distribution

There is a mathematical object, or more precisely a *distribution*, that is particularly interesting because it challenges our usual notion of what a function is. This object is known as the **Dirac delta distribution**, denoted by $\delta(t)$.

The Dirac delta is often represented graphically as an infinitely narrow spike of unit area located at the origin, while its shifted version $\delta(t-a)$ is represented by the same spike located at $t = a$.

Although it is commonly written as

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0, \end{cases}$$

this piecewise description is only an intuitive representation. More rigorously, the Dirac delta is defined through its *sifting property*

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1,$$

and by the fact that it extracts the value of a continuous function at the point where it is located.

The Dirac delta may also be viewed as the limiting case of a family of ordinary functions whose width tends to zero while their area remains equal to one. In this limiting process, the peak becomes infinitely large, the width tends to zero, and the total area remains unity.

Answer the following questions.

1. Evaluate
$$\int_{-\infty}^{\infty} \delta(t)\, f(t)g(t)\, dt.$$

2. Evaluate
$$\int_{\infty}^{\infty} \delta(t)\, f(t)\, dt.$$

3. Evaluate
$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt,$$
where $a > 0$.

4. Evaluate
$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)g(t)\, dt.$$

5. Evaluate
$$\int_{0}^{\infty} \delta(t)\, dt.$$
Justify your answer carefully by referring to the defining properties of the Dirac delta distribution. State clearly any convention that you use.

Bonus (Not Graded)

This question is completely optional and carries no marks. You are not required to attempt it. It is included purely for those who are curious and would like to explore an interesting topic.

During our discussion on recurrence relations, we saw how simple recursive equations can produce surprisingly rich behaviour. One of the most famous examples of this is the *Logistic Map*, a recurrence relation that plays a central role in the study of **Chaos Theory**. Despite its simple mathematical form, it exhibits remarkably complex and unpredictable behaviour.

The Logistic Map is given by
$$x_{n+1} = rx_n(1 - x_n),$$
where $0 \leq x_n \leq 1$ and r is a parameter.

Answer the following questions.

1. Read about the Logistic Map and briefly explain why it is considered one of the simplest examples of deterministic chaos.

2. As the parameter r increases, an interesting pattern begins to emerge in the long-term behaviour of the sequence. Describe this pattern in your own words. (Hint: Look up the term *period-doubling*.)

3. What is the **Feigenbaum Constant**? Why is it considered one of the most remarkable constants in mathematics?

For the coding enthusiasts out there: Is it possible to code this map out yourselves? Try using the help of AI as much as you want, or if you're really good, then feel free to try it on your own. Try plotting the Logistic Map for different values of r and observe how the behaviour of the sequence changes.

§4 Lecture Slides and Additional Resources

- **Lecture Slides:** `https://canva.link/veyruwvl1heo0zk`
- *Concrete Mathematics: A Foundation for Computer Science* by Ronald Graham, Donald Knuth, and Oren Patashnik — Chapters 1, 2, 6, and 7 are highly relevant for Recurrences.
- *Elementary Differential Equations and Boundary Value Problems* by William E. Boyce and Richard C. DiPrima — All chapters are relevant, contains Numerical methods as well.
- **YouTube Live Stream- 10th July:** `https://www.youtube.com/watch?v=kzzSHs66Ds0&t=8179s`
- **The Dirac Delta Function:** `https://en.wikipedia.org/wiki/Dirac_delta_function`
- **The Logistic Map:** `https://en.wikipedia.org/wiki/Logistic_map`
- **Some Common Laplace Transforms** (Search up for more): `https://www.elprocus.com/what-is-laplace-transf`