Shape of Ideas : Problem Set 1 CΦ

Mathematics Club IITM

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- *All questions from section 1 are compulsory. Even if you're unsure of the answer, write your initial thoughts, approach, or reasoning. Do not leave any question blank.*
- You must think independently and refer to credible resources if needed.
- Use of Large Language Models (LLMs) like ChatGPT, Gemini, or Copilot is strictly discouraged. If detected, the submission will be disqualified.
- Submit your answers in a single PDF file. Handwritten work is allowed but must be neatly scanned or photographed and compiled.
- Name your file as: `YourName_Problem_set_1.pdf`.
- Provide clear explanations. For theoretical questions, justify your answers. For calculations or code, include brief reasoning or method.
- Ensure the work is your own. Discussions are permitted, but plagiarism is not. Any plagiarism will lead to disqualification.
- Submit your assignment by **14 July 2025**. Late submissions will not be accepted without prior approval.
- Submission is through a Google Form. In the form, you must paste the **link to the PDF stored in your Google Drive** — do not upload the file directly to the form. Make sure the link provides access to anyone with the link.
- Attempt the bonus question section for extra points!!
- Feel free to reach out to us for doubts! Contact information of the problem-set creators:
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§1 Questions

All questions in this section are worth 3 points.

1. Give the homogeneous coordinates of the following lines:
   a. $3x + 4y = 10$
   b. $5x = 19$
   c. x-axis and y-axis

Solution:

Recall that a line in the Euclidean plane given by $ax + by + c = 0$ has homogeneous coordinates $[a : b : c]$. We rewrite the equations in this form.

a. Given: $3x + 4y = 10$

Rewrite as: $3x + 4y - 10 = 0$

So the homogeneous coordinates are:

$$[3 : 4 : -10]$$

b. Given: $5x = 19$

Rewrite as: $5x - 19 = 0$

So the homogeneous coordinates are:

$$[5 : 0 : -19]$$

c. The x-axis is given by $y = 0$, or $0x + 1y + 0 = 0$

Homogeneous coordinates:

$$[0 : 1 : 0]$$

The y-axis is given by $x = 0$, or $1x + 0y + 0 = 0$

Homogeneous coordinates:

$$[1 : 0 : 0]$$

2. Is the cross ratio a symmetric relation? Explain with the help of an example.

Solution:

The cross ratio of four collinear points A, B, C, D is defined as:

$$(A, B; C, D) = \frac{(C - A)(D - B)}{(D - A)(C - B)}$$

This expression is **not symmetric** in general. That is, permuting the points usually changes the value of the cross ratio.

Example:

Let the points be:

$$A = 0, \quad B = 1, \quad C = 2, \quad D = 3$$

Then the cross ratio is:

$$(0, 1; 2, 3) = \frac{(2 - 0)(3 - 1)}{(3 - 0)(2 - 1)} = \frac{2 \cdot 2}{3 \cdot 1} = \frac{4}{3}$$

Now reverse the last two points:

$$(0, 1; 3, 2) = \frac{(3-0)(2-1)}{(2-0)(3-1)} = \frac{3 \cdot 1}{2 \cdot 2} = \frac{3}{4}$$

Hence,

$$(0, 1; 2, 3) = \frac{4}{3} \neq \frac{3}{4} = (0, 1; 3, 2)$$

So the cross ratio is not symmetric under permutation of the points.

Conclusion:

- The cross ratio is **not symmetric** in general.
- It is invariant under projective transformations, not arbitrary point swaps.
- There are exactly six distinct values of the cross ratio that can arise from permutations of the four points.
- In special configurations (e.g., harmonic division), some symmetry may exist.

3. Given four concurrent lines, how would you compute their cross ratio using an arbitrary transversal line?

Solution:

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines passing through a common point O, i.e., they are **concurrent** at O.

To compute the cross ratio of these four lines, denoted as:

$$(\ell_1, \ell_2; \ell_3, \ell_4)$$

we proceed as follows:

Step-by-step Procedure

1. Choose an arbitrary line t (called a **transversal**) that does not pass through O, and intersects each of the four lines $\ell_1, \ell_2, \ell_3, \ell_4$ at distinct points.
2. Let the points of intersection be:

$$A = \ell_1 \cap t, \quad B = \ell_2 \cap t, \quad C = \ell_3 \cap t, \quad D = \ell_4 \cap t$$

3. Compute the cross ratio of the four points on the transversal line t:

$$(\ell_1, \ell_2; \ell_3, \ell_4) := (A, B; C, D)$$

where the right-hand side is the usual cross ratio of four collinear points, given by:

$$(A, B; C, D) = \frac{(C - A)(D - B)}{(D - A)(C - B)}$$

assuming the points are expressed in coordinate form along the transversal.

Key Properties

- The value of the cross ratio $(\ell_1, \ell_2; \ell_3, \ell_4)$ is **independent of the choice of the transversal** t. That is, any transversal will yield the same value (up to projective equivalence), provided the order of the lines is preserved.
- The cross ratio of four concurrent lines is a **projective invariant**.

4. Compute the cross ratio of each of the following sets of four collinear points on the real projective line (1-D space). Recall that the cross ratio of points A, B, C, D (assumed to lie on a line) is given by:

$$\text{CR}(A, B; C, D) = \frac{(A - C)(B - D)}{(A - D)(B - C)}$$

a) $A = 0, \quad B = 1, \quad C = 2, \quad D = 3$

b) $A = -2, \quad B = -1, \quad C = 1, \quad D = 2$

c) $A = 0, \quad B = \frac{1}{2}, \quad C = 1, \quad D = 2$

Solution:

Recall that the cross ratio of four collinear points A, B, C, D is given by:

$$\text{CR}(A, B; C, D) = \frac{(A - C)(B - D)}{(A - D)(B - C)}$$

a) **Given:** $A = 0, \quad B = 1, \quad C = 2, \quad D = 3$

$$\begin{aligned}\text{CR}(0, 1; 2, 3) &= \frac{(0 - 2)(1 - 3)}{(0 - 3)(1 - 2)} \\ &= \frac{(-2)(-2)}{(-3)(-1)} \\ &= \frac{4}{3}\end{aligned}$$

b) **Given:** $A = -2, \quad B = -1, \quad C = 1, \quad D = 2$

$$\begin{aligned} \text{CR}(-2,-1;1,2) &= \frac{(-2-1)(-1-2)}{(-2-2)(-1-1)} \\ &= \frac{(-3)(-3)}{(-4)(-2)} \\ &= \frac{9}{8} \end{aligned}$$

c) **Given:** $A = 0, \quad B = \frac{1}{2}, \quad C = 1, \quad D = 2$

$$\begin{aligned} \text{CR}\left(0, \tfrac{1}{2}; 1, 2\right) &= \frac{(0-1)\left(\frac{1}{2}-2\right)}{(0-2)\left(\frac{1}{2}-1\right)} \\ &= \frac{(-1)(-1.5)}{(-2)(-0.5)} \\ &= \frac{1.5}{1} = \frac{3}{2} \end{aligned}$$

5. Why can't projective geometry define concepts like "midpoint" or "betweenness" without affine distinctions?

Solution:

Projective geometry is concerned with properties of geometric configurations that are invariant under **projective transformations**. These include:

- Incidence (which points lie on which lines),
- Cross ratio of four collinear points,
- Concurrency of lines,
- Collinearity of points.

It does **not** rely on concepts such as:

- Distance,
- Angles,
- Parallelism,
- Ratios of lengths.

Affine Concepts Not Preserved in Projective Geometry

Concepts like **midpoint** and **betweenness** rely on *distance ratios* or a notion of measurement. Let's examine each:

1. **Midpoint:**

Let A and B be two points on a line. The midpoint M satisfies:

$$\text{Distance}(A, M) = \text{Distance}(M, B)$$

or, in affine coordinates:

$$M = \frac{1}{2}A + \frac{1}{2}B$$

This definition requires a notion of **vector addition and scalar multiplication**, which are not available in the projective setting.

Furthermore, under a general projective transformation, the image of the midpoint of a segment is **not** necessarily the midpoint of the transformed segment. Thus, midpoint is not a projective invariant.

2. **Betweenness:**

   Suppose three collinear points A, B, C satisfy that B lies *between* A and C. This implies a linear ordering or distance-based separation:

   $$\text{Distance}(A, B) + \text{Distance}(B, C) = \text{Distance}(A, C)$$

   Again, this uses distance or a metric, which projective geometry does not possess.

   Projective transformations can map lines in such a way that the order of collinear points changes, so betweenness is **not preserved** projectively.

Cross Ratio Does Not Encode Midpoint or Betweenness Alone

Although projective geometry preserves the **cross ratio** of four collinear points, the cross ratio alone cannot distinguish which point lies between the others or which point is a midpoint.

For example, the cross ratio $(A, B; C, D) = -1$ implies that C and D are harmonic conjugates with respect to A and B, but this does **not** mean that either of them is the midpoint of the segment AB. It only tells us something about their *harmonic relation*, not about equal distances.

Conclusion

Projective geometry lacks the structure needed to define concepts based on measurement or ordering, such as:

- Midpoint (requires vector addition and scalar multiplication),
- Betweenness (requires linear ordering or distance),
- Length and angles (require metric structure).

These concepts belong to affine or Euclidean geometries, which impose additional structure on the projective plane. Thus, without introducing affine distinctions, projective geometry cannot define or preserve midpoint or betweenness.

6. Lines in projective space are considered closed curves. Why?

Solution:

In projective geometry, particularly in the real projective plane $\mathbb{RP}^2$ or the projective line $\mathbb{RP}^1$, lines are treated differently than in Euclidean geometry. A fundamental reason lines are considered **closed curves** in projective geometry is the **addition of "points at infinity"** that eliminate boundaries and extend lines beyond their affine representations.

1. Affine vs Projective Line

In affine geometry (e.g., $\mathbb{R}^2$), a line extends infinitely in both directions, but it does *not* form a closed curve — it is open-ended.

In projective geometry, we extend the affine plane $\mathbb{R}^2$ to the projective plane $\mathbb{RP}^2$ by adding a **line at infinity**, consisting of **points at infinity** corresponding to each direction. In particular:

- A line in $\mathbb{RP}^2$ intersects the line at infinity in exactly one point.
- Parallel lines in $\mathbb{R}^2$ meet at a point at infinity in $\mathbb{RP}^2$.

Thus, each projective line is topologically equivalent to a **circle** (i.e., a 1-sphere S^1) — it has no beginning and no end.

2. Homogeneous Coordinates Perspective

A line in the projective plane is given by a linear equation in homogeneous coordinates:

$$ax + by + cz = 0$$

This equation includes all points $[x : y : z]$ satisfying the relation, including those where $z = 0$, i.e., points at infinity. Thus, projective lines are naturally compact and closed.

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7. Prove the principle of duality in a projective space for the axiom:
"If the six vertices of a hexagon lie alternately on two lines, the three points of intersection of pairs of opposite sides are collinear."

Solution:

Pascal's Theorem (Primal Statement)

"If the six vertices of a hexagon lie alternately on two lines, the three points of intersection of pairs of opposite sides are collinear."

This is a special case of **Pascal's Theorem**, where a hexagon is inscribed in a conic (in this case, a degenerate conic made of two lines). It is an important theorem in projective geometry.

Let the hexagon be labeled $ABCDEF$, such that:

$$A, C, E \in \ell_1 \quad \text{and} \quad B, D, F \in \ell_2$$

Then consider the three pairs of opposite sides:

$$\text{Side pair } (AB, DE) \Rightarrow \text{intersection } P$$
$$\text{Side pair } (BC, EF) \Rightarrow \text{intersection } Q$$
$$\text{Side pair } (CD, FA) \Rightarrow \text{intersection } R$$

Pascal's Theorem asserts that:

$$\text{Points } P, Q, R \text{ are collinear.}$$

Principle of Duality

In projective geometry, the principle of duality states:

> Every theorem or definition remains valid if we interchange the terms "point" and "line," and replace "lies on" with "passes through".

Dual of Pascal's Theorem (Brianchon's Theorem)

Applying duality to the above statement:

"If the six sides of a hexagon are tangent alternately to two conics (or in this degenerate case, pass through two fixed points), then the three lines connecting opposite vertices are concurrent."

In our degenerate case, we interpret the dual of Pascal's configuration with a hexagon A, B, C, D, E, F circumscribed about two fixed points (dual of two lines). Then the lines joining opposite vertices:

$$AD, BE, CF$$

are concurrent — they meet at a single point.

This is precisely the content of **Brianchon's Theorem** in degenerate form.
Proof: The primary statement involves nine points and nine lines, which can be drawn in many ways (apparently different though projectively equivalent), such as the two shown in Figure in slides. $A_1B_2C_1A_2B_1C_2$ is a hexagon whose vertices lie alternately on the two lines $A_1B_1C_1$, $A_2B_2C_2$. The points of intersection of pairs of opposite sides are

$$A_3 = B_1C_2 \cdot B_2C_1, \quad B_3 = C_1A_2 \cdot C_2A_1, \quad C_3 = A_1B_2 \cdot A_2B_1.$$

The axiom asserts that these three points are collinear. Our notation has been devised in such a way that the three points A_i, B_j, C_k are collinear whenever

$$i + j + k \equiv 0 \pmod 3).*$$

Another way to express the same result is to arrange the 9 points in the form of a matrix

$$\begin{Vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{Vmatrix}$$

If this were a determinant that we wished to evaluate, we would proceed to multiply the elements in triads. These six "diagonal" triads, as well as the first two rows of the matrix, indicate triads of collinear points. The axiom asserts that the points in the bottom row are likewise collinear. Its inherent self-duality is seen from an analogous matrix of lines

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix}.$$

These lines can be picked out in many ways, one of which is

$$\begin{aligned} a_1 &= A_3B_1C_2, & b_1 &= A_1B_3C_2, & c_1 &= A_2B_2C_2, \\ a_2 &= A_2B_3C_1, & b_2 &= A_3B_2C_1, & c_2 &= A_1B_1C_1, \\ a_3 &= A_1B_2C_3, & b_3 &= A_2B_1C_3, & c_3 &= A_3B_3C_3. \end{aligned}$$

Which completes the proof of duality for the last axiom.

8. A geometry **G** is an **affine plane** if the following three properties are satisfied:

Af1. For every pair of distinct points of **G**, there is a unique line $\ell \in \mathcal{L}_{\mathbf{G}}$ containing them.

Af2. There is a four element subset of **G** in which no three points are collinear.

Af3. For each line ℓ and point $P \notin \ell$, there is a unique line ℓ' on P parallel to ℓ.

A geometry **G** is a **projective plane** if **G** has the following three properties:

Pr1. For every pair of distinct points of **G**, there is a unique line $\ell \in \mathcal{L}_{\mathbf{G}}$ containing them.

Pr2. There is a four element subset of **G** in which no three points are collinear.

Pr3. Any two lines in **G** intersect in at least one point.

An affine plane has *order* n if every line has n points on it.

- Can you draw affine planes of order 2, 3, 4?
- Can you draw an affine plane with some lines containing 3 points and some lines containing 4 points?

Solution:

Recall the properties of an affine plane **G** of order n:

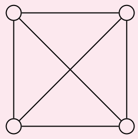
- Each line contains exactly n points.
- Through every point, there are exactly $n + 1$ lines.

- The total number of points is n^2.
- The total number of lines is $n(n+1)$.

We now construct affine planes of order 2, 3, and 4, and address whether a valid affine plane can have lines with different numbers of points.

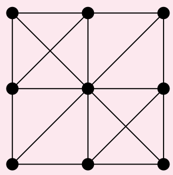
Affine Plane of Order 2

There are 4 points and 6 lines. Each line contains 2 points.



Affine Plane of Order 3

There are 9 points and 12 lines. Each line has 3 points.



Affine Plane of Order 4 (Sketch Idea)

For order 4:

- 16 points
- 20 lines
- Each line contains 4 points

Drawing is complex and often done via coordinates in $\mathbb{Z}_4 \times \mathbb{Z}_4$. It is best visualized using coordinates and modular arithmetic.

Can an Affine Plane Have Mixed Line Sizes?

No.

It would contradict the uniform incidence structure required by axioms **Af1** and **Af3**.

Why Not?

Assume some lines have 3 points, others 4. Then:

- The number of lines through a point would vary — violating **Af3**.
- The total number of points and incidences would not match any known affine order.
- Parallelism structure would break — for some lines, you could not find exactly one parallel line through a given point.

Conclusion

- Affine planes of order 2 and 3 can be explicitly drawn with lines having exactly n points.
- All valid affine planes must have **uniform line sizes**.
- An affine plane where some lines contain 3 points and others 4 is **not** possible.

9. Redo the above for projective planes.

Solution:

A projective plane is a geometry $\mathbf{G} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is the set of points and $\mathcal{L}$ the set of lines, satisfying the following axioms:

Pr1. Any two distinct points lie on exactly one line.

Pr2. Any two distinct lines meet in exactly one point.

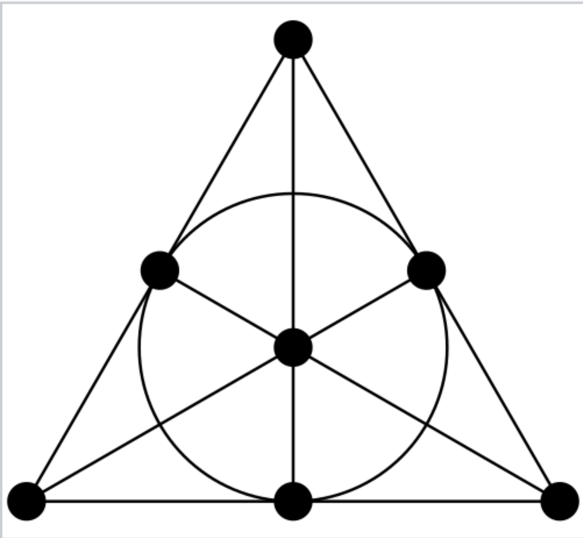
Pr3. There exists a set of four points, no three of which lie on the same line.

A projective plane is said to have **order** n if:

- Each line contains $n + 1$ points,
- Each point lies on $n + 1$ lines,
- There are $n^2 + n + 1$ points and $n^2 + n + 1$ lines in total.

Projective Plane of Order 2 (Fano Plane)

- $n = 2$
- Points: $2^2 + 2 + 1 = 7$
- Lines: 7
- Each line contains 3 points



The Fano plane

Projective Plane of Order 3

- $n = 3$
- Points: $3^2 + 3 + 1 = 13$
- Lines: 13
- Each line contains 4 points

This plane is often denoted $\mathbb{P}^2(\mathbb{F}_3)$, and its points can be represented by homogeneous coordinates $[x : y : z]$ over $\mathbb{F}_3$, excluding the zero vector and identifying scalar multiples.

Due to complexity, we do not draw this plane explicitly but note that it exists and is unique up to isomorphism.

Projective Plane of Order 4

- $n = 4$
- Points: $4^2 + 4 + 1 = 21$
- Lines: 21
- Each line contains 5 points

This projective plane can be constructed algebraically using $\mathbb{P}^2(\mathbb{F}_4)$, the projective plane over the field with 4 elements. Drawing it by hand is impractical, but the structure obeys all axioms.

Can a Projective Plane Have Mixed Line Sizes?

No.

In a valid projective plane of order n, all lines must contain exactly $n + 1$ points, and all points must lie on exactly $n + 1$ lines.

Why Not?

- The axioms require symmetry and uniformity: every pair of distinct lines must intersect at exactly one point (**Pr2**), which is not possible if line sizes vary.
- The combinatorial structure of a projective plane depends on fixed incidence parameters:

$$v = b = n^2 + n + 1, \quad r = k = n + 1$$

  where:
  - v: number of points,
  - b: number of lines,
  - r: number of lines through each point,
  - k: number of points on each line.
- If line sizes were not constant, these equations would break, and the structure would no longer be a projective plane.

Conclusion

- Projective planes of orders 2, 3, and 4 all satisfy the axioms with uniform line and point incidence.
- Each line must contain exactly $n + 1$ points.
- A projective plane with lines of mixed sizes (e.g., some with 3 points and others with 4) is not possible.

10. Show that $\mathbb{Z}_p^2$ is an affine plane for p prime, but that $\mathbb{Z}_3^2$ is neither affine nor projective.
    - How many lines are there in $\mathbb{Z}_3^2$?
    - How many lines are there in $\mathbb{Z}_p^2$?
    - How many families of parallel lines are there in each case?

Solution:

1. Affine Plane Structure of $\mathbb{Z}_p^2$

Let p be a prime number, and consider the set $\mathbb{Z}_p^2 = \{(x, y) \mid x, y \in \mathbb{Z}_p\}$.

This set consists of p^2 points. We define a line in $\mathbb{Z}_p^2$ as the set of solutions to a linear equation of the form:

$$ax + by = c$$

for fixed $a, b \in \mathbb{Z}_p$, not both zero, and $c \in \mathbb{Z}_p$.

Why $\mathbb{Z}_p^2$ Forms an Affine Plane (When p is Prime)

- Given any two distinct points in $\mathbb{Z}_p^2$, there is a unique line containing both (since $\mathbb{Z}_p$ is a field).
- For each line and a point not on it, there is a unique parallel line through that point (with same direction vector (a, b), different c).
- There exist 4 points no 3 of which are collinear — for example, the square $(0, 0), (1, 0), (0, 1), (1, 1)$.

Hence, $\mathbb{Z}_p^2$ is an affine plane of order p, when p is prime.

2. Why $\mathbb{Z}_3^2$ is Not a Projective Plane

Though $\mathbb{Z}_3$ is a field (so the basic arithmetic works), the set $\mathbb{Z}_3^2$ does not include the "points at infinity" required for projective closure.

- It only has $3^2 = 9$ points.
- A projective plane of order 3 must have $3^2 + 3 + 1 = 13$ points and lines, which $\mathbb{Z}_3^2$ does not.
- It also lacks a proper line-at-infinity to define direction classes fully.

Thus, $\mathbb{Z}_3^2$ is a grid of points but is not closed under projective incidence.

3. How Many Lines in $\mathbb{Z}_3^2$?

We list all possible directions $(a, b) \in \mathbb{Z}_3^2 \setminus \{(0, 0)\}$. Up to scalar multiples (i.e., same slope), we get:

$$\text{Directions: } (1, 0), (0, 1), (1, 1), (1, 2)$$

So, there are 4 slope (direction) classes. For each slope class, there are 3 parallel lines (one for each value of $c \in \mathbb{Z}_3$).

$$\Rightarrow 4 \text{ slope classes} \times 3 \text{ lines each} = 12 \text{ lines total}$$

4. How Many Lines in $\mathbb{Z}_p^2$?

In general:

- There are p^2 points.
- The number of slope (direction) classes is $\frac{p^2-1}{p-1} = p+1$ (each distinct direction (a, b), up to scalar multiple).
- Each direction class gives rise to p parallel lines.

So the total number of lines is:

$$(p+1) \cdot p = p^2 + p$$

5. How Many Families of Parallel Lines?

- Each distinct direction $(a : b) \in \mathbb{P}^1(\mathbb{Z}_p)$ defines a class of parallel lines.
- Number of such direction classes $= p+1$

Summary

- $\mathbb{Z}_p^2$ is an affine plane when p is prime.
- $\mathbb{Z}_3^2$ is **not** a projective plane.
- Number of lines in $\mathbb{Z}_3^2$: 12
- Number of lines in $\mathbb{Z}_p^2$: $p^2 + p$
- Number of families of parallel lines: $p+1$

§2 Bonus Question

(This section is worth for 5 points. Try to give it a fair attempt.)
Learn and tell us axioms of some other geometry which was not discussed in the session and assignments.

Solution:

The answer is context-dependent and may differ among individuals.