Shape of Ideas : Problem Set 1

Mathematics Club IITM

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General Instructions:

- *All questions from section 1 are compulsory. Even if you're unsure of the answer, write your initial thoughts, approach, or reasoning. Do not leave any question blank.*
- You must think independently and refer to credible resources if needed.
- Use of Large Language Models (LLMs) like ChatGPT, Gemini, or Copilot is strictly discouraged. If detected, the submission will be disqualified.
- Submit your answers in a single PDF file. Handwritten work is allowed but must be neatly scanned or photographed and compiled.
- Name your file as: `YourName_Problem_set_2.pdf`.
- Provide clear explanations. For theoretical questions, justify your answers. For calculations or code, include brief reasoning or method.
- Ensure the work is your own. Discussions are permitted, but plagiarism is not. Any plagiarism will lead to disqualification.
- Submit your assignment by **14 July 2025**. Late submissions will not be accepted without prior approval.
- Submission is through a Google Form. In the form, you must paste the **link to the PDF stored in your Google Drive** — do not upload the file directly to the form. Make sure the link provides access to anyone with the link.
- Attempt the bonus question section for extra points!!
- Feel free to reach out to us for doubts! Contact information of the problem-set creators:
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§1 Questions

All questions in this section are worth 3 points.

1. Give the homogeneous coordinates of the following lines:
   a. $3x + 4y = 10$
   b. $5x = 19$

c. x-axis and y-axis

2. Is the cross ratio a symmetric relation? Explain with the help of an example.

3. Given four concurrent lines, how would you compute their cross ratio using an arbitrary transversal line?

4. Compute the cross ratio of each of the following sets of four collinear points on the real projective line (1-D space). Recall that the cross ratio of points A, B, C, D (assumed to lie on a line) is given by:

$$\mathrm{CR}(A, B; C, D) = \frac{(A - C)(B - D)}{(A - D)(B - C)}$$

a) $A = 0, \quad B = 1, \quad C = 2, \quad D = 3$

b) $A = -2, \quad B = -1, \quad C = 1, \quad D = 2$

c) $A = 0, \quad B = \frac{1}{2}, \quad C = 1, \quad D = 2$

5. Why can't projective geometry define concepts like "midpoint" or "betweenness" without affine distinctions?

6. Lines in projective space are considered closed curves. Why?

7. Prove the principle of duality in a projective space for the axiom:
"If the six vertices of a hexagon lie alternately on two lines, the three points of intersection of pairs of opposite sides are collinear."

8. A geometry $\mathbf{G}$ is an **affine plane** if the following three properties are satisfied:

Af1. For every pair of distinct points of $\mathbf{G}$, there is a unique line $\ell \in \mathcal{L}_{\mathbf{G}}$ containing them.

Af2. There is a four element subset of $\mathbf{G}$ in which no three points are collinear.

Af3. For each line ℓ and point $P \notin \ell$, there is a unique line ℓ' on P parallel to ℓ.

A geometry $\mathbf{G}$ is a **projective plane** if $\mathbf{G}$ has the following three properties:

Pr1. For every pair of distinct points of $\mathbf{G}$, there is a unique line $\ell \in \mathcal{L}_{\mathbf{G}}$ containing them.

Pr2. There is a four element subset of $\mathbf{G}$ in which no three points are collinear.

Pr3. Any two lines in $\mathbf{G}$ intersect in at least one point.

An affine plane has *order* n if every line has n points on it.

- Can you draw affine planes of order 2, 3, 4?
- Can you draw an affine plane with some lines containing 3 points and some lines containing 4 points?

9. Redo the above for projective planes.

10. Show that $\mathbb{Z}_p^2$ is an affine plane for p prime, but that $\mathbb{Z}_3^2$ is neither affine nor projective.
    - How many lines are there in $\mathbb{Z}_3^2$?
    - How many lines are there in $\mathbb{Z}_p^2$?
    - How many families of parallel lines are there in each case?

§2 Bonus Question

(This section is worth for 5 points. Try to give it a fair attempt.)

Learn and tell us axioms of some other geometry which was not discussed in the session and assignments.