Shape of Ideas: Problem Set 2 CΦ

Mathematics Club IITM

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General Instructions:

1. *All questions are compulsory. Even if you're unsure of the answer, write your initial thoughts, approach, or reasoning. Do not leave any question blank.*
2. You are encouraged to think independently and refer to credible resources if needed.
3. Use of Large Language Models (LLMs) like ChatGPT, Gemini, or Copilot is strictly discouraged. If detected, the submission will be disqualified.
4. Submit your answers in a single PDF file. Handwritten work is allowed but must be neatly scanned or photographed and compiled.
5. Name your file as: `YourName_AssignmentTitle.pdf`.
6. Provide clear explanations. For theoretical questions, justify your answers. For calculations or code, include brief reasoning or method.
7. Ensure the work is your own. Discussions are permitted, but copying is not. Any plagiarism will lead to rejection.
8. Submit your assignment by **16 July 2025**. Late submissions will not be accepted without prior approval.
9. Submission is through a Google Form. In the form, you must paste the **link to the PDF stored in your Google Drive** — do not upload the file directly to the form. Make sure the link has appropriate viewing permissions.

- Feel free to reach out to us for doubts! Contact information of the problem-set creators:
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Submit here: Google Form Link

The total marks is capped at 30, the bonus points will be added if you lose points in any main question.

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§1 Questions

1. a) Prove using induction (3 marks)

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

   b) Prove the cube of any number can be written as the difference between the squares of 2 integers

2. Find $(1^p + 1)(2^p + 1)(3^p + 1)\cdots(99^p + 1) \pmod p$, where $p = 101$. (3 marks)

3. Find all pairs of positive primes p, q satisfying $p - q = 5$ (2 marks)

4. For all positive integers n, let $T_n = 2^{2^n} + 1$. Show that if $m \neq n$, then T_m and T_n are relatively prime. (4 marks)
   Hints: Subtract a quantity from T_n to obtain a neat factorisation.

5. Find the general form of solution to the following system of equation (5 marks)

$$\begin{aligned} &18x - 23y = 31 \\ &3x + 12 \equiv 17 \bmod (29) \\ &5x - 8 \equiv 22 \bmod (17) \end{aligned}$$

   Hint: You can construct solutions using the Chinese Remainder Theorem, research how to do that

6. Derived a rational approximation of $\sqrt{23}$ by using the continued fraction representation and Pell's equation. (4 marks)

7. Use theory of congruences to prove that there doesn't exists integral solutions for the equation (4 marks)

$$x^2 - y^2 = 1002.$$

   Hint: Try using small moduli to derive contradictions

8. a) Prove (5 marks)

$$n = \sum_{d|n} \phi(d)$$

   where ϕ is the Euler totient function.
   Hint: Try dividing all numbers from 1 to n into classes based on $\gcd(x, n)$.

   b) Prove

$$\phi(n) = \sum_{d|n} d\mu\left(\frac{d}{n}\right)$$

   where μ is the Möbius function.
   (This is a continuation of the above question so you may assume (a) is true)

§2 Bonus Question

1. Prove the Euler totient function $\phi(n)$ is multiplicative. (3 marks)

   Note: If you want to use the closed form expression of ϕ you need to prove that again without using the fact that the totient function is multiplicative.