Shape of Ideas : Problem Set 4

Mathematics Club IITM

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- The solutions to the questions in Problem Set 4 are given in this file.
- Feel free to reach out to us for doubts! Contact information of the problem set creator:
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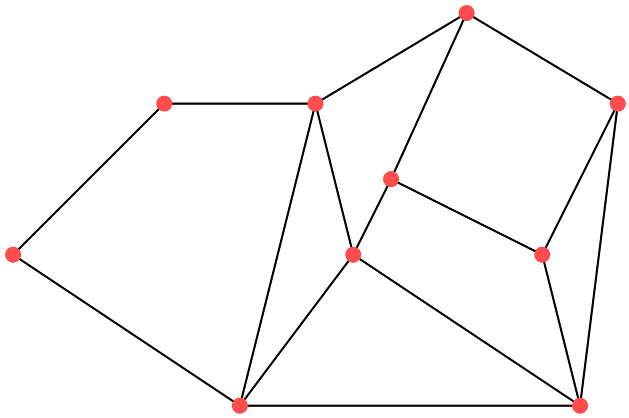
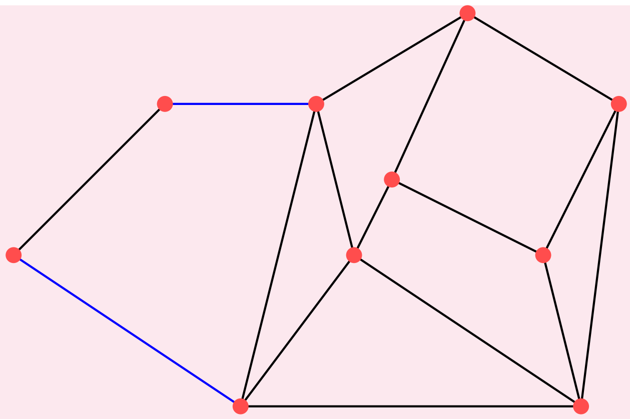
Solutions

1. For the below graph, find the:
   (a) edge connectivity
   (b) vertex connectivity

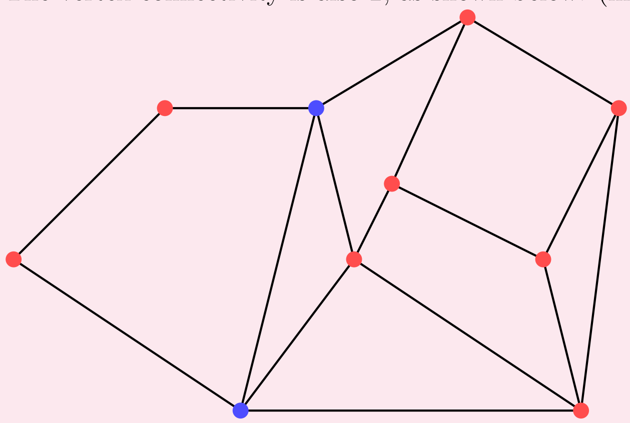


Solution:

(a) The edge connectivity is 2, as shown below: (in blue)



(b) The vertex connectivity is also 2, as shown below: (in blue)



2. Let G be a connected undirected graph with exactly four vertices of odd degree.

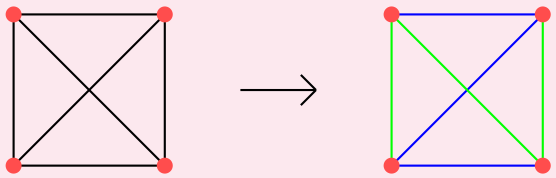
   (a) Does an Eulerian circuit exist in G? If so, give an example. If not, why?

   (b) Is it always possible to partition the edge set of such a graph into exactly two edge-disjoint trails? If yes, give an example.

Solution:

(a) No. A connected and undirected graph has an Eulerian circuit only if all its vertices have even degree.

(b) Yes. Any graph with exactly $2k$ vertices of odd degree can be partitioned into k edge-disjoint trails. In the above case, $k = 2$. An example depiction of this is as follows:



3. Given a simple undirected graph G with n vertices, suppose that for every pair of non-adjacent vertices u

and v, $\deg(u) + \deg(v) \geq n$. Prove that G contains a Hamiltonian cycle.

Solution:

Proof by contradiction:

- Assume G is not Hamiltonian. Consider a longest possible cycle C in G.
- If C is not Hamiltonian, there exists a vertex x not on C.
- Since C is maximal, x cannot be inserted into C to form a longer cycle, which leads to a contradiction using the degree condition.
- Specifically, for any two non-adjacent vertices u, v on C, the sum of their degrees is at least n, which forces the existence of edges that allow extending C, contradicting maximality.
- Therefore, G must be Hamiltonian.

4. Consider a bipartite graph H with bipartition (A, B), $|A| = |B| = n$, and suppose every vertex has degree at least $n/2$. Does H always contain
   (i) a Hamiltonian path?
   (ii) a Hamiltonian cycle? Justify your answer.

Solution:

(i) No.
The following is a counterexample:
   - Partition A and B each into two equal halves: $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, with $|A_1| = |A_2| = |B_1| = |B_2| = n/2$ (assuming n is even).
   - Connect every vertex in A_1 to every vertex in B_1, and every vertex in A_2 to every vertex in B_2.
   - Each vertex has degree $n/2$, but the graph is disconnected, so no Hamiltonian path exists.

(ii) No. The above counterexample applies here too—without a Hamiltonian path itself, there exists no question of a Hamiltonian cycle.

5. Let $n \geq 4$ be an integer, and let T be a labeled tree on the vertex set $1, 2, ..., n$.
   (a) Let $d \geq 2$. How many labeled trees are there on n vertices in which a fixed vertex, say vertex 1, has degree exactly d? Provide an explicit formula in terms of n and d.
   (b) For $n = 7$, how many labeled trees are there in which both vertex 1 and vertex 2 have degree 3?

Solution:

(a) If a vertex has degree d, it means that it must appear exactly $d - 1$ times in the Prüfer code.

For an n-vertex tree, the Prüfer code has $n-2$ elements, of which $d-1$ must be the vertex of interest i.e. vertex 1.

- First, we choose $d-1$ of the $n-2$ positions to assign to vertex 1 i.e. $\binom{n-2}{d-1}$
- Then, we fill the remaining $n-d-1$ positions with any of the $n-1$ labels except 1: for each, $n-1$ choices, and $n-d-1$ times i.e. $(n-1)^{(n-d-1)}$.

Hence, the total number of such trees will be $\boxed{\binom{n-2}{d-1} \cdot (n-1)^{(n-d-1)}}$.

(b) If $n = 7$, and both vertex 1 and vertex 2 have degree 3, then the Prüfer code will be of length 5 and feature:

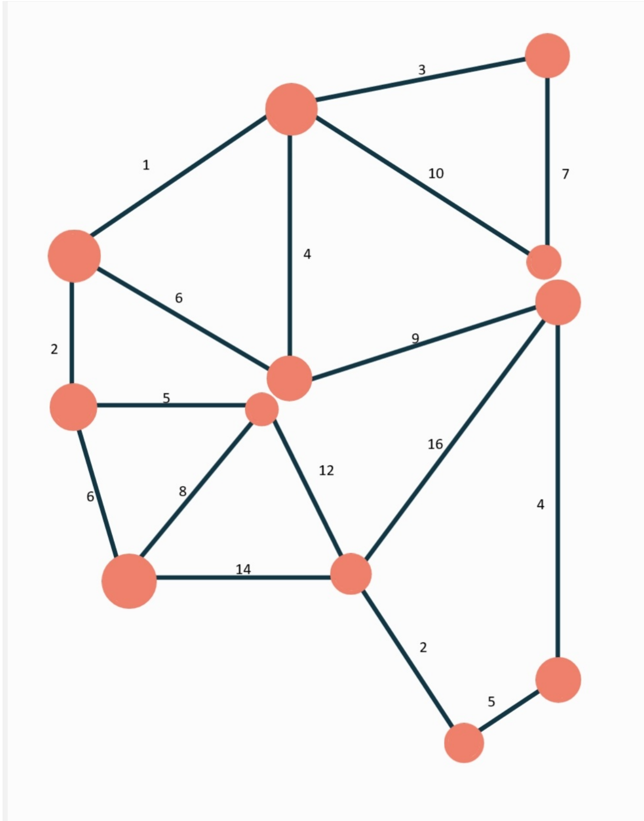
- 1 and 2 twice each; and
- any element from the set $\{3, 4, 5, 6, 7\}$ once.

Hence, we have:

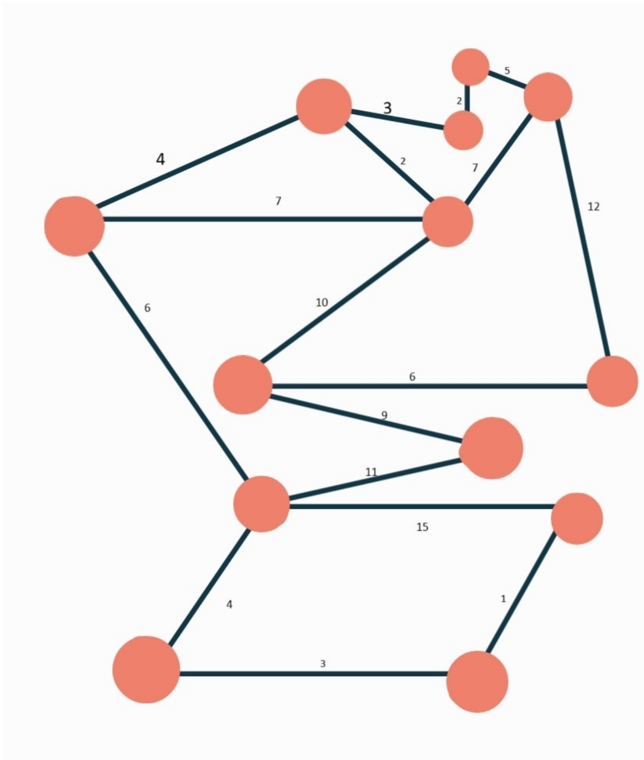
- Two positions (out of 5) for 1: $\binom{5}{2} = 10$.
- Two positions (out of remaining 3) for 2: $\binom{3}{2} = 3$.
- One position (only remaining), to be filled by one of $\{3, 4, 5, 6, 7\}$: $\binom{5}{1} = 5$.

Hence, the total number of such trees will be $10 \cdot 3 \cdot 5 = \boxed{150}$.

6. Find the minimum spanning tree of the following graphs, using one of the major algorithms. Explain the steps clearly.



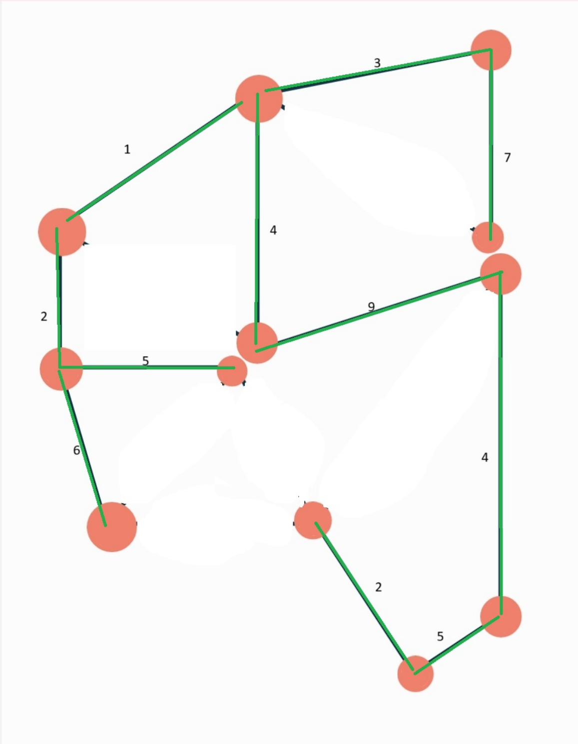
(a)



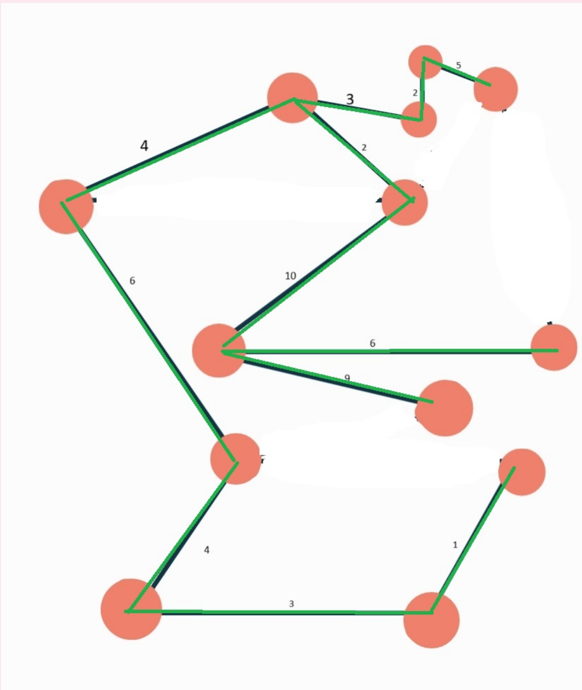
(b)

Solution:

Any of the major algorithms can be utilized to find the MST.



(a)



(b)

7. Determine the critical 3-chromatic graphs. Give your reasoning for your answer.

Solution:

The only graphs which are 3-critical (in the sense of vertex colorings) are cycles of odd length: C_{2n+1} for $n \geq 1$ (i.e., cycles with 3, 5, 7, etc., vertices).
The reasoning for this is as follows:

- If a graph G is 3-critical, it is not 2-colorable and thus contains an odd cycle.
- Let C be the smallest odd cycle in G. The subgraph induced by C must have chromatic number 3. If G contained more than just C, then C would be a proper induced subgraph with the same chromatic number as G, contradicting criticality.
- Therefore, G must be exactly the odd cycle C.

8. Prove the recursive bound of Ramsey numbers i.e. $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Solution:

- **Base cases**: $R(s,1) = R(1,t) = 1$ and $R(s,2) = s$, $R(2,t) = t$.
- Suppose the recursive bound holds for smaller values of s and t. Let $R(s-1,t) + R(s,t-1) = N$. We consider any possible red/blue coloring of the edges of the complete graph K_N.
- Pick any vertex v in K_N. v is connected by edges to all other $N-1$ vertices. Each of these edges can be red or blue. Let's partition the remaining vertices:
  - Red-neighbours : Vertices joined to v with a red edge. Let the set be A.
  - Blue-neighbours : Vertices joined to v with a blue edge. Let the set be B.
  - Note: $|A| + |B| = N-1$.
- Given the choice of N, it must be that:
  - Either $|A| \geq R(s-1,t)$; or
  - $|B| \geq R(s,t-1)$.

  This is because $|A| + |B| = N - 1 = R(s-1,t) + R(s,t-1) - 1$. If both sets were smaller than their respective Ramsey numbers, their combined size would be less than $N-1$, which is impossible by the pigeonhole principle.
- By induction:
  - **Case 1**:$|A| \geq R(s-1,t)$
    Consider the subgraph induced by A. By the definition of $R(s-1,t)$, the red-blue coloring of A's edges must contain either:
    - a red K_{s-1}: adding v (which is connected to all of A in red) gives a red K_s,
    - or a blue K_t: we are done.

- **Case 2**:$|B| \geq R(s, t-1)$
  Now consider the subgraph induced by B. By the definition of $R(s, t-1)$, the red-blue coloring of B's edges must contain either:
  * a blue K_{t-1}: adding v (which is connected to all of B in blue) gives a blue K_t,
  * or a red K_s: we are done.

In either case, the complete graph on $N = R(s-1,t) + R(s,t-1)$ vertices must contain either a red K_s or a blue K_t. Thus:$R(s,t) \leq R(s-1,t) + R(s,t-1)$

9. How does Euler's formula constrain the possible structures of planar graphs, and what does it reveal about the maximum number of edges such graphs can have in relation to their number of vertices?

Solution:

Euler's formula for connected planar graphs is: $v - e + f = 2$ where v is the number of vertices, e is the number of edges and f is the number of faces. This imposes limits on the construction of planar graphs:

- In any planar embedding, each edge borders two faces, and each face is bounded by at least three edges (for simple graphs with $v \geq 3$).
- The formula shows that planar graphs cannot be arbitrarily dense; there is an upper limit on the number of edges for a given number of vertices. This upper limit is shown below:
  - Each edge is shared by two faces, so $2e \geq 3f$.
  - Substitute f from Euler's formula: $f = 2 - v + e$.
  - Plug into the inequality: $2e \geq 3(2 - v + e)$.

  Solving for e:
  $2e \geq 6 - 3v + 3e$
  $\implies 2e - 3e \geq 6 - 3v$
  $\implies -e \geq 6 - 3v$
  $\implies e \leq 3v - 6$
  Thus, we conclude that for any simple, connected planar graph with $v \geq 3$, the number of edges e is bounded by the result $\boxed{e \leq 3v - 6}$.