Shape of Ideas : Problem Set 3 CΦ

Mathematics Club IITM

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- *All questions from section 1, 2, 3 are compulsory. Even if you're unsure of the answer, write your initial thoughts, approach, or reasoning. Do not leave any question blank.*
- You must think independently and refer to credible resources if needed.
- Use of Large Language Models (LLMs) like ChatGPT, Gemini, or Copilot is strictly discouraged. If detected, the submission will be disqualified.
- Submit your answers in a single PDF file. Handwritten work is allowed but must be neatly scanned or photographed and compiled.
- Name your file as: `YourName_Problem_set_3.pdf`.
- Provide clear explanations. For theoretical questions, justify your answers. For calculations or code, include brief reasoning or method.
- Ensure the work is your own. Discussions are permitted, but plagiarism is not. Any plagiarism will lead to disqualification.
- Submit your assignment by **19 July 2025**. Late submissions will not be accepted without prior approval.
- Submission is through a Google Form. In the form, you must upload your assignment in the appropriate place. Note that the assignment should in the PDF format and should not exceed 10MB.
- Feel free to reach out to us for doubts! Contact information of the problem-set creators:
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§1 Level 1(Very Easy)

All questions in this section are worth* 1 *points.

1. In a city, no two people have an identical set of teeth, and there is no person without a tooth. Also, no person has more than 31 teeth. If we disregard the shape and size of tooth and consider only the positioning of the teeth, what is the maximum population of the city?
2. What is the number of numbers which are less than 1000 that can be formed using the digits $0, 1, 2, 3, 4, 5$ when repetition is not allowed?
3. What is the total number of five-digit numbers of different digits in which the digit in the middle is the largest?
4. Find the 7^{th} Catalan number C_7.
5. In how many ways can you represent 20 as a sum of four positive integers?

§2 Level 2 (Moderate)

All questions in this section are worth* 2 *points.

1. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then find the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$.
2. A debate club consists of 6 girls and 5 boys. A team of 5 members is to be selected from this club including the selection of a captain (from among these 5 members) for the team. If the team has to include at most one boy, then what is the number of ways of selecting the team?
3. In how many ways can three persons A, B, C having respectively $6, 7$ and 8 one-rupee coins donate 13 rupees collectively?
4. How many ways are there to triangulate a convex hexagon into 4 triangles by drawing non-intersecting diagonals?
5. In how many ways can you represent 15 as a sum of three positive integers such that each of them is greater than 2?

§3 Level 3(Hard)

All questions in this section are worth 3 points.

1. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point and the loser got 0 points, and each of the two players earned 1/2 point if the game was a tie. After completion of the game, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten). What is the total number of players in the tournament?

2. Let n be an odd integer greater than 1. Prove that the sequence $\binom{n}{1}, \binom{n}{2}, ..., \binom{n}{\frac{n-1}{2}}$ contains an odd number of odd numbers.

3. 23 people of positive integral weights decide to play football. They select one person as the referee and then split up into two 11-person teams of equal total weights. It turns out that no matter who the referee is this can always be done. Prove that all 23 people have equal weights

4. Consider a Dyck path of length $2n$, which consists of n up steps $(1, 1)$ and n down steps $(1, -1)$, starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x axis. Each up step can be colored either red or blue, with red steps having weight 2 and blue steps having weight 1. The weight of a Dyck path is the product of the weights of its up steps. Find the sum of the weights of all Dyck paths of length $2n$, and express the answer in terms of Catalan numbers. Verify for $n = 2$.

5. In how many ways can you represent 10 as a sum of four positive integers such that each integer is strictly less than 6?