SRMC 2025 Prelims Questions

Mathematics Club IITM × Maths and Physics Club IITB

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§1 Questions and Solutions

§1.1 A simple rounding question

Let $f(n)$ be the closest integer to $n^{1/2}$. Find the value of $\sum_{n=1}^{2025} \frac{1}{f(n)}$

Solution:

For a given k such that $\left(k - \frac{1}{2}\right) \leq n^{1/2} \leq \left(k + \frac{1}{2}\right)$, $f(n)$ will be equal to k. The range of n values for a given k is $\left(k + \frac{1}{2}\right)^2 - \left(k - \frac{1}{2}\right)^2 = 2k$. Thus the sum reduces to $\left(\sum_{k=1}^{44} \frac{2k}{k}\right) + 1 = 89$

Answer: 89

§1.2 Patten

Consider the infinite sequence:

$$S = 1\ 2\ 2\ 3\ 3\ 3\ 4\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 6\ 6\ \ldots$$

in which each natural number n appears exactly n times when written in ascending order as shown above (note that when $n = 10$, 1 and 0 are considered two separate digits in the sequence when written down).
Let N be the product of all the non zero digits that occur between the 10,000$^{\text{th}}$ and the 11,071$^{\text{st}}$ digit (inclusive) of the sequence.
Compute:

$$\log_6 N$$

Solution:

$$\text{Number of 1-digit numbers: } 1 + 2 + \cdots + 9 = 45$$
$$\text{Number of 2-digit numbers: } 10 + 11 + \cdots + 99 = 4905$$
$$\text{Total digits from 1- and 2-digit numbers: } 45 + 2 \times 4905 = 9855$$

Hence the 10,000$^{\text{th}}$ and 11,071$^{\text{st}}$ digits are at the starting of the 3 digit numbers.

Each 3-digit number n appears n times, each time as 3 digits. Hence we can add 300, 303, 306 etc to 9855 in order to get their last appearance.

100 is between 9856 and 10155, 101 is between 10156 and 10458, 102 is between 10459 and 10764 and 103 is between 10765 and 11073. Since 0's and 1's do not matter in the product, we can ignore the 100's and 101's in our range. Our range ends at 11071 while 103 ends at 11073, hence we will have every single 2 from 102's and every single 3 from 103's except for the last one. Hence 2 and 3 both appear exactly 102 times, making N 6^{102}, and our final answer as 102.

§1.3 Takeshi's Castle

Pratyaksh is standing on an arbitrary point on the lines (walls) of the hexagonal setup given below.

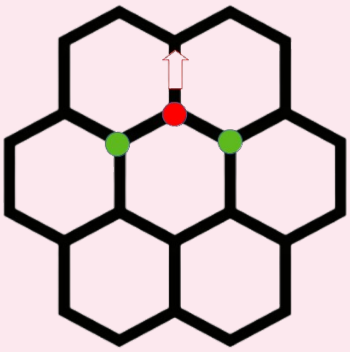
Takeshi and his team of n guards (also starting from an arbitrary position) is trying to catch him. Takeshi and his guards can move as fast as Pratyaksh. Find the minimum number of guards (excluding Takeshi himself) needed.

Solution:

Answer: 1

It may seem like 3 people are needed since a node has max 3 edges, but actually 2 people (including Takeshi) are enough:

Step 1: Force Pratyaksh out of the centre Hexagon, i.e surround him from two directions to force him into the third.



Let us denote Takeshi and his guards with *green* and Pratyaksh with *red*.

Step 2: Force Pratyaksh into a corner, i.e a node with only two lines. One of the green guards move towards Pratyaksh and one replaces the red vertex.

Pratyaksh is either forced left or right.

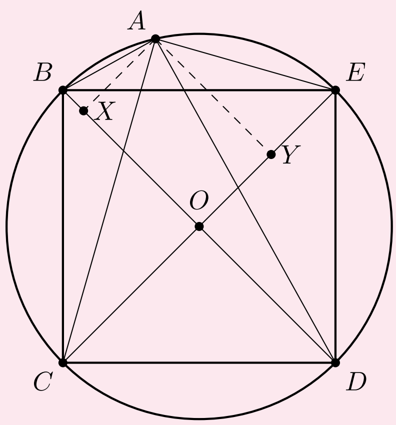
Step 3: Pratyaksh is surrounded before he can escape the corner. This is possible because the green point is at a distance of 2 units to the exit while the red point is at a distance ≥ 2 units (depending on how far he ran while being approached in step 2) at the start of step 3.

§1.4 Square in a Circle

A, B, C, D, E are five points on a circle with $AB \times AD = 56$ and $AC \times AE = 90$. If $BCDE$ is a square, find its area.

Solution:

Let O be the centre and r be the radius of the circle. Since $BCDE$ is a square, BD and CE meet at O. Let X and Y be the feet of perpendiculars from A to BD and CE respectively. Then $AXOY$ is a rectangle.
Note that $\triangle ABD$ is right-angled at A.



Considering its area gives $BD \cdot AX = AB \cdot AD$, so that $2r \cdot AX = 224$.
Similarly, we have $2r \cdot AY = 360$.
On the other hand, since $AX^2 + AY^2 = XY^2 = OA^2 = r^2$, we have the equation

$$\left(\frac{56}{2r}\right)^2 + \left(\frac{90}{2r}\right)^2 = r^2.$$

$\implies r^2 = \sqrt{28^2 + 45^2} = 53.$
And so the area of the square $BCDE$ is $2r^2 = 106$.

Answer: 106

§1.5 Not the collatz conjecture

There is a positive integer 'n' written on a board. If that number is even, it is replaced by $\frac{n}{2}$ else it is replaced by $n^2 - 1$. Once 0 appears on the board we terminate this series of operations. Suppose the initial number is of the form $2^n \pm 2^m (n \geq m$ and $n, m \in Z_{\geq 0})$. Find the number of pairs (m, n) where $0 \leq m < n \leq 10$ for which this sequence will terminate.

Solution:

It will terminate for all n, m because after dividing it by 2 we reach $2^{n-m} \pm 1$ and the squaring it and subtracting 1 from it, we are left with a number which can be divided by 2^{n-m+1} so we get the new number as $2^{n-m-1} \pm 1$

Answer: 55

§1.6 Magic Matrix

Rhythm constructs magic matrices where a magic matrix $M = [a_{ij}]_{3\times3}$ is made up of non-zero numbers such that every row and every column of the matrix sums up to 2025. If the value of $\Delta = k_{max} - k_{min}$ (k_{max} and k_{min} are the maximum and minimum values attained by k over all the possible magic matrices he can construct) where

$$k = \sum_{i,j=1}^{3} \left(\prod_{k=1}^{3} a_{ik} - \prod_{k=1}^{3} a_{kj} \right)$$

then find the value of 2025Δ.

Solution:

Consider the equality:

$$\sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij} \right)^3 = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} a_{ij} \right)^3 . \quad (*)$$

Both sides involve the cubes of all elements a_{ij}. The coefficients for a_{11}^2 on the left and right sides are equal to $3(a_{21} + a_{31})$ and $3(a_{12} + a_{13})$, respectively. These numbers are equal because $a_{11} + a_{12} + a_{13} = a_{11} + a_{21} + a_{31}$. Similarly, the coefficients of $a_{12}^2, \ldots, a_{33}^2$ are also equal. Therefore, from the equality $(*)$ follows the equality

$$6 \sum_{i=1}^{3} \prod_{j=1}^{3} a_{ij} = 6 \sum_{i=1}^{3} \prod_{j=1}^{3} a_{ji}.$$

This leads to the final equality:

$$6a_{11}a_{21}a_{31} + 6a_{12}a_{22}a_{32} + 6a_{13}a_{23}a_{33} = 6a_{11}a_{12}a_{13} + 6a_{21}a_{22}a_{23} + 6a_{31}a_{32}a_{33}.$$

Thus $k_{max} = k_{min} = 0 \implies \Delta = 0$

Answer: 0

§1.7 Digits

Find the number of pairs (M, N) of numbers M and N where:

- all of the digits of M are even and all of the digits of N are odd,
- and each digit from 0 to 9 occurs exactly once among M and N,
- and N divides M.

Solution:

Answer : $\boxed{0}$
First of all we must observe that both M and N are 5 digit numbers. N is a permutation of the digits $(1, 3, 5, 7, 9)$ and M is a permutation of the digits $(2, 4, 6, 8, 0)$ ensuring that 0 is not at the start (from the left) . Now define the function $S(P)$ as the sum of digits of P .
Now $S(M) = 2 + 4 + 6 + 8 + 0 = 20 \equiv 2 (\text{mod } 9) \implies M \equiv 2 \pmod 9$
Similarly $S(N) = 1 + 3 + 5 + 7 + 9 = 25 \equiv 7 \pmod 9 \implies N \equiv 7 \pmod 9$
This gives us that $2 \equiv 7k \pmod 9$, where $k = \dfrac{M}{N}$
$\therefore k \equiv 2 \cdot 7^{-1} \pmod 9 \equiv 8 (\text{mod } 9)$.
Thus he minimum value of k is 8. But if we multiply 8 with N we will definitely get a 6 digit number(The smallest value of N is 13579, and $8N$ has 6 digits for this value on N). But M is a 5 digit number. Therefore no such pairs of numbers exist. So the answer is $\boxed{0}$.

§1.8 Latin squares

How many ways are there to fill a 4×4 matrix with the numbers 1 to 4 such that each row and each column contains exactly one of each number?

Solution:

- Fix the first row as $(1, 2, 3, 4)$ WLOG. Multiply the final count by $4! = 24$.
- Fix the first element of the second row as 2 WLOG. Multiply later by 3 (choices: 2, 3, or 4).
- Now consider two cases for the second row:
  - **Case 1:** Second row starts has 2nd element as 1. Then the row must be 2 1 4 3. In this case, the 3rd row, 1st 2 elements can be 3 4 or 4 3 and next to can be 1 2 or 2 1. Hence $2 \times\ 2 = 4$ ways.
  - **Case 2:** 2nd row has 2nd element as 3 or 4(2 choices). Each one of those choices gives one of 2 cyclic permutations for the 3rd row , hence total $2 \times\ 2 = 4$ ways.
  - In either case, 4th row is always fixed after fixing 3rd row.
- Total for fixed first row and second row starting with 2: $4 + 4 = 8$
- Multiply: 24 (first rows) $\times$ 3 (choices for second row start) $\times\ 8 = \boxed{576}$

§1.9 Trapped in the Trapezium

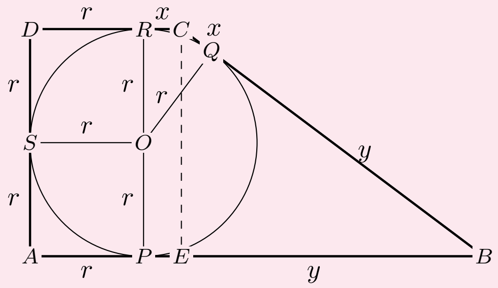
$ABCD$ is a trapezium with $AB \parallel CD$ and $AB \perp AD$. Given that $AB = 3 \cdot CD$ and the Area of $\square ABCD = 16$, find the square of the radius of a circle that can be drawn to touch all sides $ABCD$. If it is not possible to construct such a circle, put your answer as -1.

Solution:

Consider the circle (of radius r) to touch the sides of the trapezium AB, BC, CD, DA at P, Q, R, S respectively.

Using Tangent from an external point theorem:

AP = AS = r (= OP)
BQ = BP = y (let)
CR = CQ = x (let)
DS = DR = r (= OR)



This gives us that:

$$AB = r + y, BC = x + y, CD = r + x, AD = 2r$$

Given $AB = 3 \cdot CD \implies r + y = 3 \cdot (r + x) \implies y = 2r + 3x$

$$\text{Using } \square ABCD = \frac{1}{2} \cdot AD \cdot (AB + CD) \implies \frac{1}{2} \cdot 2r \cdot 4(r + x) = 16$$
$$\implies r \cdot (r + x) = 4 \quad (1)$$

$$\text{Applying Pythagoras theorem in } \triangle CEB \implies CE^2 + EB^2 = CB^2$$
$$(2r)^2 + (y - x)^2 = (y + x)^2 \implies r^2 = xy$$
$$\implies r^2 = x \cdot (2r + 3x) \implies (r - 3x) \cdot (r + x) = 0$$
$$\implies r = 3x$$

Substituting in equation (1), we get

$$\boxed{r = \sqrt{3}, r^2 = 3}$$

§1.10 Living on the edge of predictability

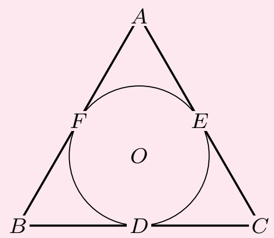
In $\triangle ABC$, the incircle touches BC, CA and AB at D, E and F respectively. If the radius of the incircle is 4 and if BD, CE, AF are consecutive integers, then find the length of side AC.

Solution:

Let $BD = x$, then $CE = x + 1$, $AF = x + 2$

Using Tangent from an external point theorem:

BF = BD = x
CD = CE = x+1
AE = AF = x+2



This gives us:

$$AB = 2x + 2 = c, BC = 2x + 1 = a, CA = 2x + 3 = b$$

$$s = \frac{(a + b + c)}{2} = 3 \cdot (x + 1)$$

$$r = \frac{\Delta}{s} = \sqrt{\frac{(s - a) \cdot (s - b) \cdot (s - c)}{s}}$$

$$\implies 4 = \sqrt{\frac{(x + 2) \cdot (x) \cdot \cancel{(x + 1)}}{3 \cdot \cancel{(x + 1)}}}$$

$$\implies 48 = x \cdot (x + 2) \implies (x + 8) \cdot (x - 6) = 0 \implies x = 6$$

$$\boxed{AC = b = 15}$$

Note: If the consecutive integers are taken to be in descending order then $\boxed{\text{AC} = \text{b} = 13}$. Both answers have been awarded marks.

§1.11 Count It!

How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and no three consecutive 1s?

Solution:

After a 0, I can have either 10 or 110 after it. This happens because after 0, I cannot place another 0 so the digit 1 has to follow 0.
I can either have another 1 after the first 1, or 0 after the first 1 And after 11, only 0 can come after it since 3 consecutive 1s are not allowed.

10 and 110 are sequences of size 2 and 3 respectively that start with 1 and end with 0.
After this 0, I can put 10 or 110 again.

Now after the first zero, I have to fill up 18 spots with either 10 or 110 series, so the size of the series is either 2 or 3 and 18 places are to be filled. So this question boils down to "how many ways can I add 2s and 3s to make the sum 18

Case 1 : (2+2+2…+2) 9 times, There is only 1 way to arrange this.
Case 2 : (3+3)+(2+2+2+2+2+2), there are 8C2 = 28 ways to do this.
Case 3 : (3+3+3+3)+(2+2+2), there are 7C3 = 35 ways for this.
Case 4 : (3+3+…3) 6 times there is only 1 way to arrange this.

Summing these numbers gives the answer as $\boxed{\mathbf{65}}$

§1.12 Double Count it!

Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $x_i \neq y_j \; \forall \, i, j \in \{1, 2, \cdots, n\}$. Define the matrix $A = (a_{ij})$ for $1 \leq i, j \leq n$, where

$$a_{ij} = \begin{cases} 2025, & \text{if } x_i > y_j \\ -2025, & \text{otherwise} \end{cases}$$

Let B be a matrix of the same dimensions as A, where $b_{ij} \in \mathbb{Z} \cap [-2025, 2025]$ for all i, j , such that the sum of the elements in each row and each column of B is equal to the corresponding row and column sum of A.
Determine the number of such matrices B.

Solution:

Let us consider the sum $S = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - y_j)(a_{ij} - b_{ij})$. Let us find the value of this sum in two ways.

1. $S = \left(\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} b_{ij}\right) - \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} b_{ij}\right) \right)$. The sum $\sum_{j=1}^{n} a_{ij}$ is nothing but the sum of

elements of a row/column which is equal to that of B. Therefore $S = 0$.

2. Let us consider two subcases ,
   a) When $x_i > y_j$, $a_{ij} = 2025$ and $2025 - b_{ij} \geq 0$ as $b_{ij} \in [-2025, 2025] \implies (x_i - y_j)(a_{ij} - b_{ij}) \geq 0$.
   b) When $x_i < y_j$, $a_{ij} = -2025$ and $-2025 - b_{ij} \leq 0$ as $b_{ij} \in [-2025, 2025] \implies (x_i - y_j)(a_{ij} - b_{ij}) \geq 0$.

   The above cases imply that $(x_i - y_j)(a_{ij} - b_{ij}) \geq 0 \;\forall\; i, j$ and the sum of all terms is zero which implies that $(x_i - y_j)(a_{ij} - b_{ij}) = 0 \;\forall\; i, j \implies a_{ij} = b_{ij} \;\forall\; i, j$

So the number of possible matrices with the constraint is $\boxed{1}$.

§1.13 2 eqns

How many triplets of integers x, y, z satisfy

$$x^2 = yz + 4 \text{ and } y^2 = zx + 4.$$

Solution:

Rename x,y,z to a,b,c :) Subtracting the 2 equations gives

$$a^2 - b^2 = c(b - a),$$

$$(a - b)(a + b + c) = 0. \tag{iii}$$

Case I: $a = b$.
Then from (i) $a^2 = ac + 4$, i.e.

$$a^2 - ac - 4 = 0.$$

Its discriminant is

$$D = c^2 + 16 = \ell^2, \quad \implies \quad \ell^2 - c^2 = 16.$$

Thus $(\ell - c)(\ell + c) = 16$. Examining integer factor pairs of 16 yields the following possibilities for (a, b, c):

a	b	c
2	2	0
−2	−2	0
−1	−1	3
4	4	3
1	1	−3
−4	−4	−3

Case II: $a + b + c = 0$ and $a \neq b$.
Then upon adding the given 2 equations and putting this condition,

$$a^2 + b^2 + c^2 = 8.$$

Checking all integer triples with sum zero and sum of squares 8 gives

a	b	c
-2	2	0
2	-2	0
0	2	-2
0	-2	2
2	0	-2
-2	0	2

Hence in total there are $6+6=\boxed{12}$ integer solutions (a,b,c).

§1.14 Think Complex

Let
$$f(x)=\int_0^x\left(1+\frac{t^6}{6!}+\frac{t^{12}}{12!}+\cdots\right)dt.$$
Find the value of $f\left(\frac{\pi}{\sqrt{3}}\right)$ with the precision of two digits, and give the answer as $100\times f\left(\frac{\pi}{\sqrt{3}}\right)$.
You may use the approximations:
$$\cosh\left(\frac{\pi}{\sqrt{3}}\right)\approx\sqrt{8},\quad \cosh\left(\frac{\pi}{2\sqrt{3}}\right)\approx 1.44.$$
Do not substitute in any approximations until you have an explicit expression for $f\left(\frac{\pi}{\sqrt{3}}\right)$
Recall the definitions:
$$\cosh(x)=\frac{e^x+e^{-x}}{2},\qquad \sinh(x)=\frac{e^x-e^{-x}}{2}.$$

Solution:

Let $\alpha=e^{i\pi/3}$. Consider the sum of exponentials:
$$\sum_{k=0}^{5}e^{\alpha^k t}$$
Expanding this sum yields:
$$\sum_{k=0}^{5}e^{\alpha^k t}=6\left(1+\frac{t^6}{6!}+\frac{t^{12}}{12!}+\cdots\right)$$
That is:
$$\sum_{k=0}^{5}e^{\alpha^k t}=6\sum_{n=0}^{\infty}\frac{t^{6n}}{(6n)!}$$
Now, integrating both sides from 0 to x:
$$\sum_{k=0}^{5}\int_0^x e^{\alpha^k t}\,dt=6\int_0^x\sum_{n=0}^{\infty}\frac{t^{6n}}{(6n)!}\,dt=6f(x)$$
Evaluating the left-hand side:

$$f(x) = \sum_{k=0}^{5} \left(\frac{e^{\alpha^k x} - 1}{6\alpha^k} \right)$$

This simplifies to a real-valued expression:

$$f(x) = \frac{1}{6}\left(e^x - e^{-x} + 2e^{x/2}\cos\left(\frac{\sqrt{3}x}{2} - \frac{\pi}{3}\right) + 2e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2} - \frac{2\pi}{3}\right)\right)$$

Evaluating at $x = \frac{\pi}{\sqrt{3}}$:

$$f\left(\frac{\pi}{\sqrt{3}}\right) = \frac{1}{6}\left(2\sinh\left(\frac{\pi}{\sqrt{3}}\right) + 2e^{\frac{\pi}{2\sqrt{3}}}\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + 2e^{-\frac{\pi}{2\sqrt{3}}}\cos\left(\frac{\pi}{2} - \frac{2\pi}{3}\right)\right)$$

As, $\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

We have:

$$f\left(\frac{\pi}{\sqrt{3}}\right) = \frac{1}{6}\left(2\sinh\left(\frac{\pi}{\sqrt{3}}\right) + 2\sqrt{3}\left(e^{\frac{\pi}{2\sqrt{3}}} + e^{-\frac{\pi}{2\sqrt{3}}}\right)\cdot\frac{1}{2}\right)$$

$$= \frac{1}{6}\left(2\sinh\left(\frac{\pi}{\sqrt{3}}\right) + 2\sqrt{3}\cosh\left(\frac{\pi}{2\sqrt{3}}\right)\right)$$

$$= \frac{1}{3}\sinh\left(\frac{\pi}{\sqrt{3}}\right) + \frac{\sqrt{3}}{3}\cosh\left(\frac{\pi}{2\sqrt{3}}\right)$$

Now, using the identity:

$$\cosh^2\left(\frac{\pi}{\sqrt{3}}\right) - 1 = \sinh^2\left(\frac{\pi}{\sqrt{3}}\right)$$

From the given values,

$$\cosh^2\left(\frac{\pi}{\sqrt{3}}\right) \approx 8, \quad \cosh^2\left(\frac{\pi}{\sqrt{3}}\right) - 1 = 7 = \sinh^2\left(\frac{\pi}{\sqrt{3}}\right) \implies \sinh\left(\frac{\pi}{\sqrt{3}}\right) \approx 2.65$$

$$f\left(\frac{\pi}{\sqrt{3}}\right) = \frac{2.65}{3} + \frac{\cosh\left(\frac{\pi}{2\sqrt{3}}\right)}{\sqrt{3}} = 0.8833 + \frac{\sqrt{3}\cosh\left(\frac{\pi}{2\sqrt{3}}\right)}{3}$$

Using the approximations and the given values, We get:

$$f\left(\frac{\pi}{\sqrt{3}}\right) \approx 0.88 + 1.732 \times 0.48 \approx 0.88 + 0.83136 = 1.71136 \approx 1.71$$

The final answer is **171**

Note: If you do not use the approximations provided, you can still arrive at 183 as the answer using alternative methods. Both 171 and 183 have been considered to be correct answers.

§1.15 Interesting Geo qn

Pranjal has a collection of many triangles that follows this amazing property: For each triangle in this amazing collection, (call its sides a, b, and c), we can indefinitely keep constructing a triangle with sides $s-a$, $s-b$, and $s-c$ where s is the semiperimeter of the triangle from which you are constructing.

For example, lets do the operations mentioned on the triangle with sides 6,7,9 and check if it can belong to Pranjal's collection: 6,7,9 ($s = 11$) → 2,4,5 ($s = 5.5$)→ 3.5, 1.5, 0.5 (not a valid triangle). So, this triangle cannot be present in Pranjal's collection.

Pranjal picks a triangle at random from his collection. Whats the probability that the triangle he is holding is isosceles? If the probability as a percentage rounded off to the closest integer is $N\%$, then report N as the answer.

Solution:

Answer: N = 100 (Only equilateral triangles are in Pranjal's collection!)

The perimeter of each new triangle constructed by the process is $(s-a)+(s-b)+(s-c) = 3s-(a+b+c) = 3s-2s = s$, that is, it is halved. Consider a new equivalent process in which a similar triangle with sidelengths $2(s-a)$, $2(s-b)$, $2(s-c)$ is constructed, so the perimeter is kept invariant.

Suppose without loss of generality that $a \le b \le c$. Then $2(s-c) \le 2(s-b) \le 2(s-a)$, and the difference between the largest side and the smallest side changes from $c-a$ to

$$2(s-a) - 2(s-c) = 2(c-a),$$

that is, it doubles. Therefore, if $c-a > 0$ then eventually this difference becomes larger than $a+b+c$, and it's immediate that a triangle cannot be constructed with the sidelengths. Hence the only possibility is

$$c - a = 0 \Longrightarrow a = b = c,$$

and it is clear that equilateral triangles can yield an infinite process, because all generated triangles are equilateral.

§1.16 Chessboard chaos

Consider a 10×10 chessboard. It is known that the maximum number of non-attacking queens that can be placed on the board is 10. For each possible arrangement of 10 non-attacking queens, construct a 10×10 matrix A such that $A_{i,j} = 1$ if there is a queen in the i-th row and j-th column, and $A_{i,j} = 0$ otherwise.
What is the sum of the determinants of all such matrices A corresponding to all possible non-attacking queen arrangements on the 10×10 chessboard? That is, compute

$$\sum_A \det(A),$$

where the sum is taken over all 10×10 matrices A as described above.
(A queen is non attacking if there is no other queen in its same row, column, or either diagonal)

Solution:

Since no 2 queens can have the same row or column, Each matrix such formed is a permutation matrix, a matrix where each row and column has exactly one 1. For such a matrix, lets try to calculate its determinant. If we start at any element, we multiply by either 1 or -1 and then are left with a smaller such matrix. Hence the determinant will be multiplied by 1 or -1 for each element giving the final determinant as either 1 or -1.

Consider a reflection of a valid matrix. It is also a valid chessboard arrangement and it can't be equal to the initial matrix. Now when we find determinant, we label each element with alternating +'s and -'s. Since every index is visited 1 time, we can see that an element contributing +1 or -1 in the end depends on the number of elements before it. If there are an odd number of elements before it, the sign it would have contributed from the initial labeling will change. So final determinant depends on the parity of this sum of number of elements before it(also called number of inversions).

We can see that upon reflecting, both the sum's should add up to 45 since each element of the sum is replaced by 9-itself. Hence it is not possible for both of them to be the same parity, resulting in each reflection cancelling itself and a total sum of $\boxed{0}$.

§1.17 Power Play

Rhythm has introduced a new scoring system for scoring numbers via iterated exponentiation, denoted by Θ_n, where for an n-digit number $A_1A_2\ldots A_n$ (A_1, A_2, ..., A_n are the digits of the number in base 10), the score is calculated as:

$$\Theta_n(A_1A_2\ldots A_n) = A_1^{A_2^{A_3^{\cdot^{\cdot^{A_n}}}}}.$$

Let S_1 be the set of all nine-digit numbers that minimize the value of $\Theta_9(A_1A_2\ldots A_9)$, and let S_2 be the set of all nine-digit numbers that maximize the value of $\Theta_9(A_1A_2\ldots A_9)$. Find the value of Δ, where

$$\Delta = \sum_{x\in S_2} x - \sum_{x\in S_1} 1.$$

Solution:

$\boxed{\text{Answer: } 837608688}$

Set S_1 (minimizing the score): Valid numbers are the ones in which no two adjacent digits are both equal to zero.

1. $A_1 = 1$: Let a_n be the number of valid numbers of length n ending in 0, and b_n the number of valid numbers ending in a nonzero digit. Then:

$$a_n = b_{n-1}, \quad b_n = 9(a_{n-1} + b_{n-1})$$

with base cases $a_1 = 1,\ b_1 = 9$. Computing up to $n = 8$ (since the first digit is fixed), we get:

$$a_8 + b_8 = 93684519$$

2. $A_2 = 0,\ A_1 \neq 1$: To avoid adjacent zeros, $A_1,\ A_3 \neq 0$. So the prefix $A_1A_2A_3$ has $8 \times 1 \times 9 = 72$ valid choices. The remaining digits $A_4, \ldots, A_9$ must form a 6-digit number with no adjacent zeros, i.e.,

$$a_6 + b_6 = 954261$$

Hence, total valid numbers: $72 \times 954261 = 68706792$

Total number of elements in the set S_1 are $= 93684519 + 68706792 = \boxed{162391311}$

Set S_2 (maximizing the score): The maximum score is all digits are 9, thus $S_2 = \{999999999\}$

Thus $\Delta = 999999999 - 162391311 = \boxed{837608688}$

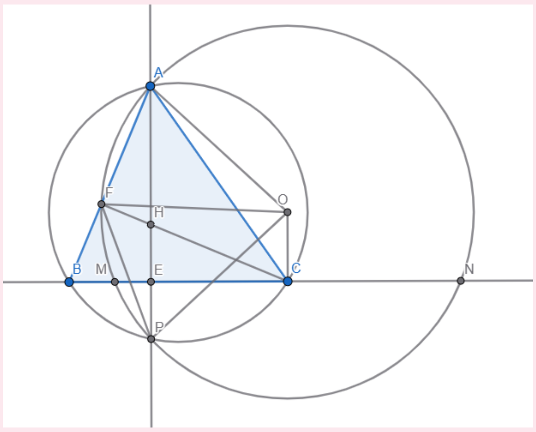
§1.18 Another interesting Geo qn

Let H be the orthocenter of an acute triangle ABC. Let F be the foot of perpendicular from C to AB. Let E be the foot of the altitude from A to BC. Let P be that point on line AH such that $PE \cdot AE = BE \cdot CE$ and

$PA > AE$. Suppose the circumcircle of triangle AFP **intersects line** BC at 2 distinct point M and N.Given that $AM = 13$, $MN = 14$ and $AN = 15$, find the value of $\lfloor AC \rfloor$ where $\lfloor x \rfloor$ denotes the largest greatest less than or equal to x.

Solution:

Note : I apologise to everyone for the incomplete question, the intersects line BC was missing in the question, without which the question does not make sense. If you're still interested in the solution, here you go.
Let $\angle AFP$ be ω. It follows that $\angle FBC$ is also ω. Notice that P lies on the circumcircle on triangle ABC, and that is the reflection of H in BC. Therefore it follows that $\angle BCP$ is also ω. Let O be the circumcenter of AFP. It is obvious that $\angle FOP = 2 \cdot \angle FAP = 2\omega$, and since $OF = OP$, $\angle FPO = 90 - \omega$. Also since $\angle FOP = \angle FCP = 2\omega$, $FOCP$ is cyclic. This means that $\angle FCO = 90 - \omega$, and therefore we get that $\angle OCB = 90$, which implies that C is the midpoint of MN, making AC the median of the triangle AMN. A simple calculation would then lead to the answer, $AC = \sqrt{148}$, which gives $\lfloor AC \rfloor = \boxed{12}$.



§1.19 Sadness

Aryan is very scared of the number 2025, because all his seniors have scared him by telling him how difficult his third semester will be. To feel happy, he thinks of the year 2027, when he can enjoy life as a fourthie. He then comes up with the following problem :

Let $S = \sum_{k=1}^{m-1} \left\lfloor \frac{k^m - k}{m} \right\rfloor$, where $m = 2027$. Find the remainder when S is divided by 2027.

Solution:

Answer : $\boxed{1014}$.
Note : All the modulos in the solution are mod 2027. Also we can drop the floors since 2027 is a prime (by Fermat's Little Theorem).
It is well known that $\sum_{k=1}^{m-1} k = \frac{m(m-1)}{2}$. Let us call this sum S_1. It thus follows that $\frac{S_1}{m} \equiv \frac{m-1}{2}$.
Using Lifting the Exponent(LTE), we have the following for all primes m and all integers $k \leq m$ ($v_p(x)$ represents the highest power of p that divides x:

$$v_m(k^m + (m-k)^m) = v_m(k + m - k) + v_m(m) = 2$$

This means that $m^2 \mid k^m + (m-k)^m$ for all primes m. Therefore summing the above relation from $k = 1$ to

$k = \frac{m-1}{2}$, we get that :

$$m^2 \;\Big|\; \sum_{k=1}^{k=m-1} k^m$$

Let us the sum on the left hand side S_2. Therefore it is clear that $\frac{S_2}{m} \equiv 0$. The given sum in the question is nothing but $\frac{S_2 - S_1}{m}$, which is congruent to $\frac{m+1}{2}$ modulo m. Substituting $m = 2027$, (2027 is a prime), we get $\boxed{1014}$.

§1.20 Linear algebra or Number Theory?

Let $A \in \mathbb{R}^{a \times m}$ and $B \in \mathbb{R}^{n \times b}$. For all A, B there is a matrix $X \in \mathbb{R}^{m \times n}$ such that

$$X \neq \mathcal{O} \quad \text{but} \quad AXB = \mathcal{O}$$

Find the number of ordered pairs (m, n) for which no pair (a, b) exists satisfying the above relation and $a^b \geq 100$ where $a, b, m, n \in \mathbb{Z}^+$.

Solution:

The given matrix relation represents a set of ab linear equations in mn variables, hence from the given matrix relation, we get $mn \geq ab$. The minimum value of ab for the inequality is 14, so $mn \leq 13$

Answer: 37

§1.21 Set partitions

Consider the sequence of integers from 1 to 103. Determine the sum of all elements in this sequence which, when removed individually, allow the remaining 102 numbers to be partitioned into 3 subsets of equal size and equal sum.

Solution:

Since 103 is 1 mod 3, the sum of all elements is 1 mod 3 as well(can be seen by analyzing $\frac{n(n+1)}{2}$). So to partition it into 3 sets of equal sum, the removed element must be 1 mod 3, as we want the sum of remaining elements to be divisible by 3.

Now we prove that any 1 mod 3 element can be removed and we can get a valid partition. After the element is removed, divide the remaining numbers into groups of 6 consecutively(102 is divisible by 6), and for each one add the 1st and 6th to the 1st subset, 2nd and 5th to the 2nd subset and 3rd and 4th to the 3rd subset. All 3 subsets will always have an equal sum in this way.

For all groups of 6 that do not contain the removed element, it is easy to see why the sum is same. Now since our removed element is 1 mod 3 and we are taking groups of 6, the element will either be in the middle of a group or in between groups. For example if 4 is removed our 1st group is $(1, 2, 3, 5, 6, 7)$ and if 7 is removed our first group is 1-6 and 2nd group is 8-13. In both cases we can see it doesn't affect division into subsets.

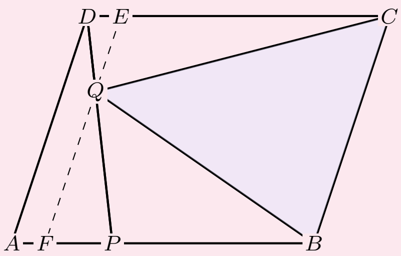
Hence the answer is 1+4+7...+103 which equals 1820.

Answer: 1820

§1.22 A Parallelogram Puzzle

$ABCD$ is a given parallelogram. A point P is taken on AB and another point Q on DP such that $AP = \frac{AB}{3}$ and $DQ = \frac{DP}{3}$. Find the ratio of areas of $\triangle QBC$ and $\square ABCD$. If the ratio in the simplest form is $\frac{m}{n}$ (here m, n are co-prime positive integers) then report $m + n$ as your answer.

Solution:



From base and height properties of area of triangle:

$$AP = \frac{AB}{3} \Rightarrow \triangle ADP = \frac{1}{3} \cdot \triangle ADB$$
$$DQ = \frac{DP}{3} \Rightarrow \triangle ADQ = \frac{1}{3} \cdot \triangle ADP$$
$$\triangle ABD = \frac{1}{2} \cdot \square ABCD \Rightarrow \triangle ADQ = \frac{1}{18} \cdot \square ABCD$$

Now, Construct $EF \parallel AD$ such that EF passes through Q.

$$\square AFED = 2 \cdot \triangle ADQ = \frac{1}{9} \cdot \square ABCD$$
$$\square BFEC = \square ABCD - \square AFED = \frac{8}{9} \cdot \square ABCD$$
$$\triangle QBC = \frac{1}{2} \cdot \square BFEC = \frac{4}{9} \cdot \square ABCD$$

$$\boxed{\frac{\triangle QBC}{\square ABCD} = \frac{4}{9}}$$

Answer: 13

§1.23 You wud rather want to skip this question

$\mathbb{Z}^+$ denotes the set of positive integers. Let a class of functions $\mathscr{F}$ be defined as follows. In this class any function $f \in \mathscr{F}$ is such that $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $m^2 + f(m)f(n)$ is divisible by $f(m) + n$, $\forall\ m, n \in \mathbb{Z}^+$. What is the minimum value of $f(100)$ where $f \in \mathscr{F}$?

Solution:

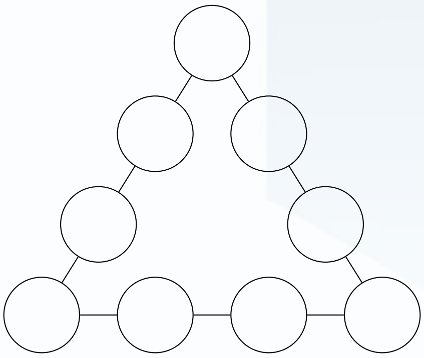
The answer is the only function in this class is $f(n) = n$ for all positive integers n.
Clearly, $f(n) = n$ for all $n \in \mathbb{Z}^+$ satisfies the original relation. We show some possible approaches to prove that this is the only possible function.
Solution. First we perform the following substitutions on the original relation:

1. With $a = b = 1$, we find that $f(1) + 1 \mid f(1)^2 + 1$, which implies $f(1) = 1$.
2. With $a = 1$, we find that $b + 1 \mid f(b) + 1$. In particular, $b \leq f(b)$ for all $b \in \mathbb{Z}^+$.
3. With $b = 1$, we find that $f(a) + 1 \mid a^2 + f(a)$, and thus $f(a) + 1 \mid a^2 - 1$. In particular, $f(a) \leq a^2 - 2$ for all $a \geq 2$.

Now, let p be any odd prime. Substituting $a = p$ and $b = f(p)$ in the original relation, we find that $2f(p) | p^2 + f(p)f(f(p))$. Therefore, the possible values of $f(p)$ are $1, p$ and p^2. By (2) above, $f(p) \geq p$ and by (3) above $f(p) \leq p^2 - 2$. So $f(p) = p$ for all primes p.
Substituting $a = p$ into the original relation, we find that $b + p \mid p^2 + pf(b)$. However, since $(b + p)(f(b) + p - b) = p^2 - b^2 + bf(b) + pf(b)$, we then have $b + p \mid bf(b) - b^2$. Thus, for any fixed b this holds for arbitrarily large primes p and therefore we must have $bf(b) - b^2 = 0$, or $f(b) = b$, as desired $\Rightarrow$ Answer: 100

§1.24 Puzzle or Proof?!

Pranjal is given the following arrangement of circles:



He has to put each number in the set $\{1, 2, \ldots, 9\}$ into one of these circles. (Each circle contains exactly 1 of the numbers). Further he has to make sure that,

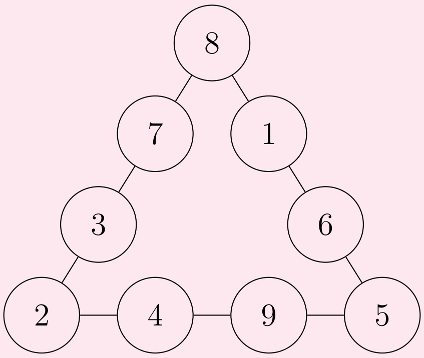
(i) the sums of the four numbers on each side of the triangle are equal;

(ii) the sums of squares of the four numbers on each side of the triangle are equal.

How many possible ways can Pranjal do so?

Solution:

The only solutions are



and the ones generated by permuting the vertices, adjusting sides and exchanging the two middle numbers on each side.

Explanation

Let a, b, and c be the numbers in the vertices of the triangular arrangement. Let s be the sum of the numbers on each side and t be the sum of the squares of the numbers on each side.
Summing the numbers (or their squares) on the three sides repeats each once the numbers on the vertices (or their squares):

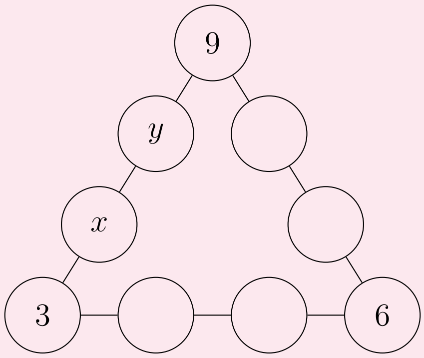
$$3s = a + b + c + (1 + 2 + \cdots + 9) = a + b + c + 45$$
$$3t = a^2 + b^2 + c^2 + (1^2 + 2^2 + \cdots + 9^2) = a^2 + b^2 + c^2 + 285$$

At any rate, $a + b + c$ and $a^2 + b^2 + c^2$ are both multiples of 3. Since $x^2 \equiv 0, 1 \pmod 3$, either a, b, c are all multiples of 3 or none is a multiple of 3. If two of them are $1, 2 \pmod 3$ then $a + b + c \equiv 0 \pmod 3$ implies that the other should be a multiple of 3, which is not possible.
Thus a, b, c are all congruent modulo 3, that is,

$$\{a, b, c\} = \{3, 6, 9\}, \quad \{1, 4, 7\}, \quad \text{or} \quad \{2, 5, 8\}.$$

Case 1: $\{a, b, c\} = \{3, 6, 9\}$. Then

$$3t = 3^2 + 6^2 + 9^2 + 285 \iff t = 137.$$



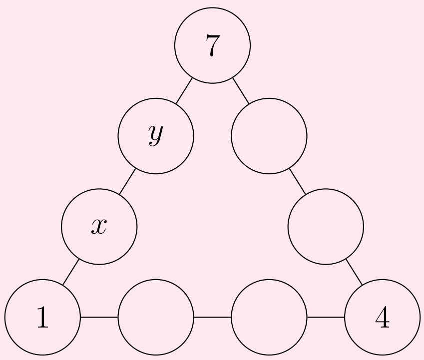
In this case

$$x^2 + y^2 + 3^2 + 9^2 = 137 \iff x^2 + y^2 = 47.$$

However, 47 cannot be written as the sum of two squares. One can check manually, or realize that $47 \equiv 3 \pmod 4$, and since $x^2, y^2 \equiv 0, 1 \pmod 4$, we must have $x^2 + y^2 \equiv 0, 1, 2 \pmod 4$, so it can never equal 47. Hence there are no solutions in this case.

Case 2: $\{a, b, c\} = \{1, 4, 7\}$. Then

$$3t = 1^2 + 4^2 + 7^2 + 285 \iff t = 117.$$



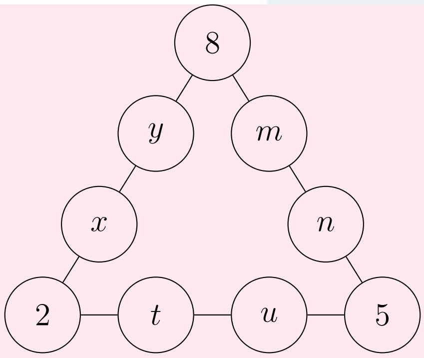
In this case

$$x^2 + y^2 + 1^2 + 7^2 = 117 \iff x^2 + y^2 \equiv 67 \equiv 3 \pmod 4,$$

and, as in the previous case, no solutions exist.

Case 3: $\{a, b, c\} = \{2, 5, 8\}$. Then

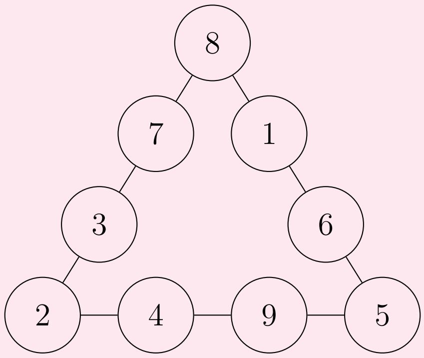
$$3t = 2^2 + 5^2 + 8^2 + 285 \iff t = 126.$$



Then

$$\begin{cases} x^2 + y^2 + 2^2 + 8^2 = 126, \\ t^2 + u^2 + 2^2 + 5^2 = 126, \\ m^2 + n^2 + 5^2 + 8^2 = 126, \end{cases} \iff \begin{cases} x^2 + y^2 = 58, \\ t^2 + u^2 = 97, \\ m^2 + n^2 = 37. \end{cases}$$

The only solutions to $t^2 + u^2 = 97$ and $m^2 + n^2 = 37$ are $\{t, u\} = \{4, 9\}$ and $\{m, n\} = \{1, 6\}$, respectively (again, one can check manually). Consequently, $\{x, y\} = \{3, 7\}$, and the solutions are



and the ones generated by permuting the vertices, adjusting sides, and exchanging the two middle numbers on each side. There are $3! \cdot 2^3 = 48$ such solutions.

Answer: 48

§1.25 Homogenous Expression, Maximization

Navin was playing around with algebraic bounds and he came up with the following expression where $x, y, z \in \mathbb{R}^+$:

$$\frac{6(x^2 + 2y^2 + 3z^2)}{6x^2 - 4xy + 14y^2 + 15z^2} + \frac{2(x^2 + 2y^2 + 3z^2)^2}{x^4 + x^2 - 4xy + 4y^2 + 4y^4 + 9z^4 + 4x^2y^2 + 6x^2z^2 + 12y^2z^2}$$

If M is the maximum value of the expression, then find $5M$.

Solution:

Notice that all the expressions involved are homogeneous and of degree 0. Set $Q := x^2 + 2y^2 + 3z^2 = 1$ WLOG (since the function due to its homogenous nature attains all values on any projective surface encompassing all directions in $\mathbb{R}^3$) and simplify each term by collecting Q's from the expression:

$$\frac{6Q}{(x - 2y)^2 + 5Q} + \frac{2Q^2}{(x - 2y)^4 + Q^2} \to \frac{6}{(x - 2y)^2 + 5} + \frac{2}{(x - 2y)^4 + 1}$$

Now, the quadratic and biquadratic in the denominators can achieve a minimum of 0. It then follows that, the terms reduce to $\frac{6}{5} + \frac{2}{1}$ at $x = 2y$. Thus the maximum is $\frac{16}{5} \implies$ Answer: 16

§1.26 Two to the Power

Rhythm has thought of n distinct numbers. He claims that for any pair of these numbers you pick, either their sum or their difference (or possibly both) will always result in a positive integral power of 2. What is the maximum number of integers he may have thought of?

Solution:

Answer: 5

Consider the set $\{-1, 1, 3, 5, 7\} \implies n \geq 5$. (Since it satisfies the condition Rhythm mentions)
To show $n > 5$ is impossible, consider the following:
At most one multiple of 3 can be on the blackboard, as the sum and difference of multiples of 3 cannot be powers of 2. Thus $n > 5 \implies 5$ non-multiples of 3.
We consider these two cases:

1. When four of them are of the form $3k+1$ (or $3k+2$):

   Let $a, b, c, d \equiv 1 \pmod 3$. Since difference of any 2 numbers is being divisible by 3, the sums $S_1 = a+b,\quad S_2 = c+d,\quad S_3 = a+c,\quad S_4 = b+d$ should be powers of 2. But $S_1 + S_2 = S_3 + S_4$ leads to a contradiction given $S_1 \neq S_3$ or S_4.

2. When three integers are of the form $3k+1$ and the other two of the form of $3k+2$. (same if it is the other way round):

   Say, $a, b, c \equiv 1 \pmod 3$ and $d, e \equiv 2 \pmod 3$. As before, each of $a+b$, $a+c$, and $b+c$ is a power of 2. Note that d cannot be larger than two of a, b, c, for if $d > a$ and $d > b$ then $d-a$, $d-b$ are powers of 2 (as $d+a$ and $d+b$ are divisible by 3), so $S_1 = d-a$, $S_2 = a+c$, $S_3 = d-b$, and $S_4 = b+c$ are all powers of 2 with $S_1 + S_2 = S_3 + S_4$. As $S_1 \neq S_3$, we must have $S_1 = S_4$, which however is impossible since $S_1 \equiv 1 \pmod 3$ while $S_4 \equiv 2 \pmod 3$.

   Hence d is larger than at most one of a, b, c, and hence smaller than at least two of them, say, $d < a$ and $d < b$. Then $S_1 = b+c$, $S_2 = a-d$, $S_3 = a+c$, and $S_4 = b-d$ are all powers of 2 with $S_1 + S_2 = S_3 + S_4$. As $S_1 \neq S_3$, we must have $S_1 = S_4$, or $c = -d$. Hence c is negative and d is positive (for otherwise we also have $d < c$ so replacing b by c in the above argument would give $b = -d$ which contradicts $b \neq c$), and hence a and b are positive. But then replacing d by e in the above argument would give $c = -e$ which contradicts $d \neq e$.

Therefore, $n \leq 5$, and the greatest possible value is $\boxed{5}$.

§1.27 The Ruin's Geometric Challenge

In an ancient ruin, Rhythm finds a triangle with angles 36°, 18°, and 126°. Besides the triangle lies a stone saying

"The key to unlocking the secrets of the ruin lies in the reflections of this points".

He thinks of an interesting activity to get the key. He selects two of these points, and draws a perpendicular bisector to the segment connecting them, and then reflects the third point relative to this perpendicular bisector to obtain a fourth point D. Rhythm continues to repeat the same procedure with the resulting set of point after each operation: selects 2 points, draws a perpendicular bisector, and all points are reflected relative to it.

What is the greatest number of different points Rhythm can obtain as result of multiple repetitions of this procedure?

Solution:

Answer: 10

Claim 1: All points lie on the circumcircle of ΔABC.

Proof. The perpendicular bisector of any chord of a circle passes through the center, and is its axis of symmetry. Therefore, any new points obtained through the given operation must lie on the circumcircle, since each operation reflects points across axes of symmetry. □

Claim 2: If the maximum number of points is reached, the resulting points form a regular polygon.

Proof. Suppose the maximum number of points has been obtained. From Claim 1, all points lie on a single circle. Label these points counterclockwise as $A_1, A_2, \ldots, A_n$. Consider three consecutive points A_{i-1}, A_i, A_{i+1}. Draw the perpendicular bisector of $A_{i-1}A_{i+1}$. If the bisector does not pass through A_i, reflecting A_i over it would produce a new point on the arc $A_{i-1}A_{i+1}$, contradicting maximality. Thus, A_i must lie on the bisector, meaning $A_iA_{i-1} = A_iA_{i+1}$, so all segments between adjacent points are equal. This implies the points form a regular polygon. □

Claim 3: The degree measure of the arc between two adjacent points cannot be less than $36°$.

Proof. Suppose the image of point Z after reflection over the perpendicular bisector of segment XY is Z', as shown in the diagram. The arcs $\widehat{Z'X}$ and $\widehat{ZY}$ are equal, and for any point T on the circle, the arc $\widehat{TZ'}$ satisfies:

$$\widehat{TZ'} = \left|\widehat{TX} \pm \widehat{YZ}\right|.$$

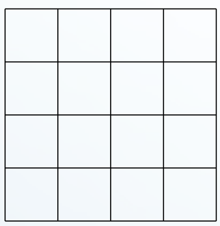
Thus, $\widehat{TZ'}$ is a multiple of the greatest common divisor (GCD) of the degree measures of arcs $\widehat{TX}$ and $\widehat{YZ}$. Initially, the arcs formed by points A, B, C are $36°, 72°, 252°$. The GCD of these values is $36°$, meaning all resulting arcs are multiples of $36°$, and hence cannot be less than $36°$. □

Answer: The total number of points are $\boxed{10}$.
From claim 3, it follows that this cannot continue indefinitely and all arcs with ends at the resulting points are multiples of 36°, so the total number of points will not be more than 10. And claim 2 tells that when the maximum is reached, the points will form a regular polygon, and the size of the arc between its adjacent vertices will be the greatest common divisor of all the remaining arcs, i.e., it will be equal to 36°. This means that there will be exactly $360°/36° = 10$ points.

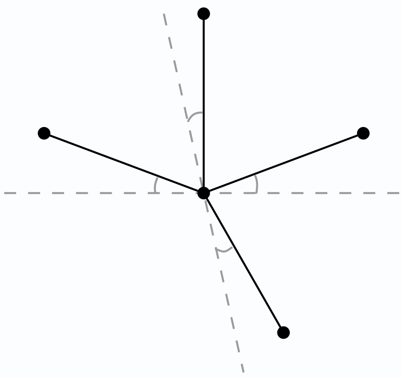
§1.28 Wrapping up

You have a square shaped wireframe grid, an $n \times n$ arrangement of points (nodes) connected by rigid straight rods along horizontal and vertical lines. The structure lies flat on a table initially and has no faces, only edges and joints — like a sieve or a mesh of rods. Each rod is rigid and cannot be bent, but a rod can bend w.r.t its node by an angle of maximum $\frac{\pi}{8}$ before it breaks.

Your goal is to bend the mesh upon itself in such a way that at least 2 distinct nodes overlap(kind of like a cylindrical roll). Find the minimum value of n which allows you to do so.



A wireframe grid for $n = 5$



The rods can be bent at a node like this with a maximum angle of $\frac{\pi}{8}$

Solution:

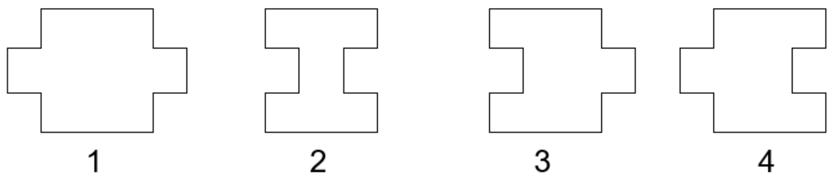
Answer: 7

One edge can bend at an angle of 22.5° w.r.t its node, so two adjacent nodes can be bent at an angle of 45° w.r.t each other. But two diagonally adjacent nodes can be bent at an angle of more than this. We can calculate that angle using cosine rule of spherical geometry.

The angle would be $arccos(cos^2 \frac{\pi}{4})$ which is $\frac{\pi}{3}$
So points 0,0 and 1,1 can be bent at an angle of 60 degrees w.r.t each other. So instead of overlapping two edges of the mesh by making a perfect cylinder, we can overlap two diagonally opposite corners, we can make 0,0 overlap with 6,6 which would give $n = 7$

§1.29 Another Puzzle

Aryan and his friend Yash are attempting to solve a puzzle, when they start thinking about the number of ways to complete the puzzle. Given below are the pieces of the puzzle. Given that there are **8** pieces of the first kind, **9** of the second kind, **10** of the 3rd kind and **11** of the 4th kind, find the number of ways that they can assemble the pieces to finish the puzzle. A puzzle is called complete if all the elements are used and adjacent pieces fit into each other perfectly. Note that all pieces of the same type are indistinguishable from each other and that you cannot rotate the elements. Also note that a solution from left to right is not the same as its reverse from right to left. They must be counted separately.
Please enter the maximum value of k, where 2^k divides the number of ways of completing the puzzle.



Solution:

Answer : 2
Proof : First we make the following simple observations. 1 could be followed by 2 or 3, 2 could be followed by 1 or 4, 3 could be followed by 2 or 3, and 4 could be followed by 1 or 4. Now notice that the only types that occur twice in a row are 3 and 4. What this motivates us to do is to merge all adjacent 3's and 4's into 3 or 4. So now we can visualise the sequence in the following way : Consider the sequence of alternating 1's and 2's starting with the 2 since it is more than the number of 1's. Now it becomes a classic balls and boxes problem, where we push to

the 3's into the boxes before each 2, and the 4's into the boxes before each 1 and the one after the last 2. The 4's therefore have $8 + 1 = 9$ boxes to go into and this can be done $\binom{9+11-1}{9-1} = \binom{19}{8}$. The 3's have 9 boxes to go into, which an be done in $\binom{9+10-1}{9-1} = \binom{18}{8}$, meaning the total number of ways is $\binom{19}{8}\binom{18}{8}$, and we can easily calculate the highest power of 2 in each using Legendre's, which comes out to be $\boxed{2}$.

§1.30 Number Game

Aryan and Abhikalp are playing a game. Given a starting number n, in each of their turns, they subtract from the ***current*** number, one of its divisors, which isn't equal to 1 or the number itself. The game ends when no move can be made, and the player whose turn it is when this happen loses. For how many values of n from 1 to 2025, does Aryan win, given that both players play optimally and Aryan plays first? Note again that they remove the divisor from the current number, and the new number post subtraction becomes the current number for the next turn.

Solution:

Answer : $\boxed{1007}$

Proof : The values of n for which Aryan wins are all even numbers except $(2, 8, 32, 128, 512)$, the odd powers of 2. We prove this in the following way :

First realise that the game ends when the number is prime or 1, For it to be 1, the previous number would have to be 2, meaning that the game ends when we hit a prime, p. Let us call the number before we hit a prime m, where $m = dk$, where d is the divisor we chose and $k, d \geq 2$. So we have $p = d(k-1)$.Now notice that means k must be 2, as d can't be 1. So $d = p$ and $k = 2$. Therefore $m = 2p$. So if players hits $2p$ for some prime p, they can just chose p as the divisor and win the game. With this mind, let us break the problem down into cases :

First consider the case when n is even but not a power of 2, which means that it has an odd divisor which doesn't equal 1 or n. If Aryan chooses this divisor, then the next number will be odd, forcing Abhikalp to choose an odd divisor, since odd numbers can't have odd divisors, ensuring that the number in the 3rd turn when it's Aryan chance again, is even.Note that it is fairly obvious that this even number also can't be a power of 2(because it is divisible by the odd divisor Abhikap chose).So Aryan just keeps choosing odd divisors on his turn. Also realise that the game must terminate in a finite number of moves, since n decreases monotonically. This means that someone must hit $2p$ eventually. But since Abhikalp never hits an even number of his turn, he can't be the person to hit $2p$, meaning that Aryan wins.

Next consider the case where n is odd. Here Aryan is forced to choose an odd divisor, so the number becomes even and is not a power of 2, since it is divisible by the odd divisor that Aryan chose. Now Abhikalp can follow the strategy of the previous case and win. This means that if n is odd, Abhilakp will win.

Now we are left with the case where n is a power of 2. Let $n = 2^m$. Now if Aryan chooses any divisor other than 2^{m-1}, then the next number would be even and have an odd divisor, allowing Abhikap to win. Therefore the optimal move is to choose the divisor 2^{m-1}, which makes the next number 2^{m-1}. Now similarly, Abhikalp must also choose half of this number or else he allows Aryan to win, and so the number keeps on being divided by 2, until we hit 2, in which case the person loses. So if the number is an even power of 2, Aryan wins and Abhilkalp wins otherwise.

Therefore Aryan wins for all even numbers except the odd powers of 2, and these are $1012 - 5 = \boxed{1007}$.