A FRIDAY EVENING ON LAGRANGE AND WEAK FORMULATIONS

Sudhanva & Arjun

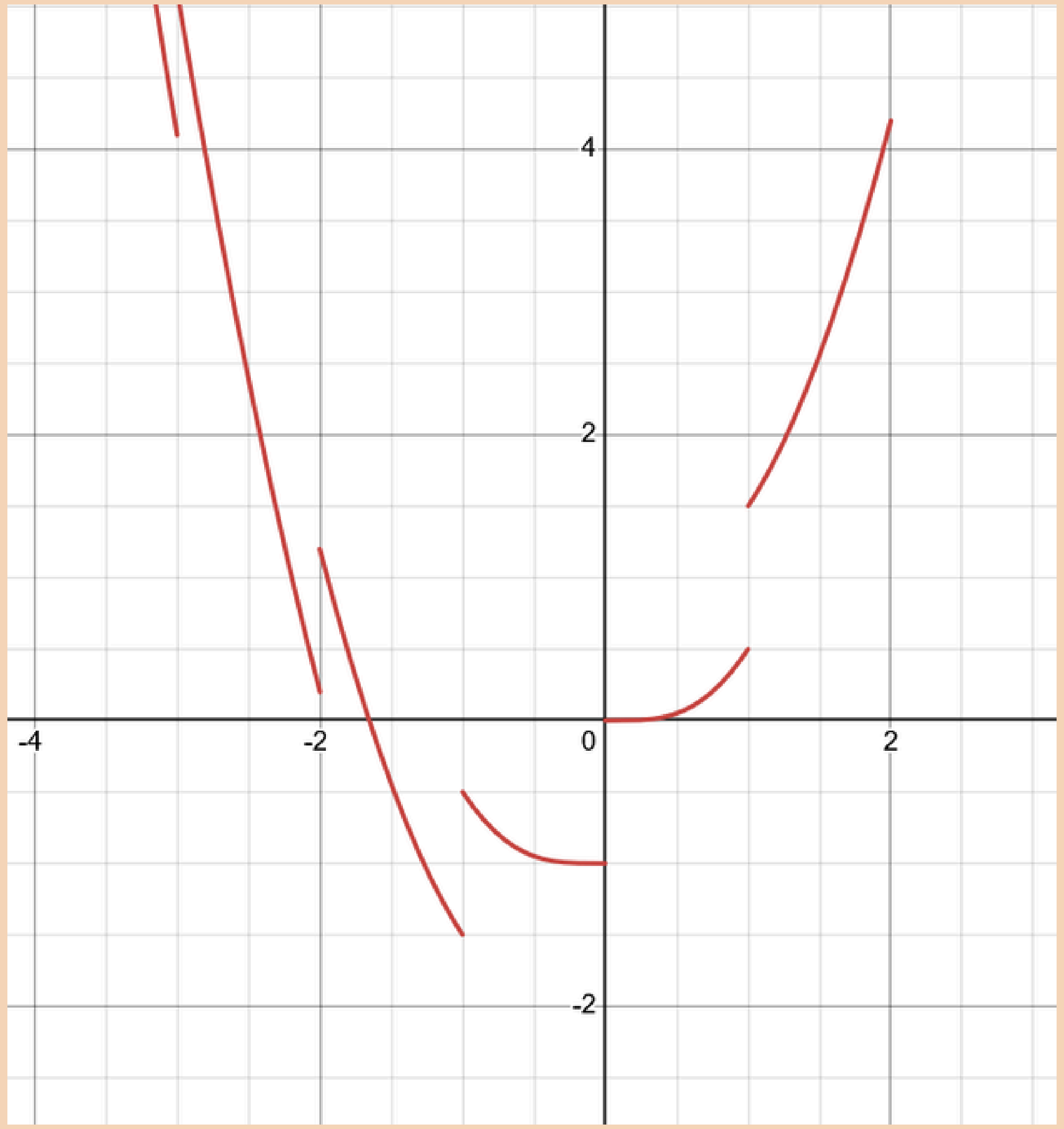
SOLVE THIS GNG

$$\frac{du}{dx} = x^2 - 1 + \frac{1}{x^2 + 1}$$

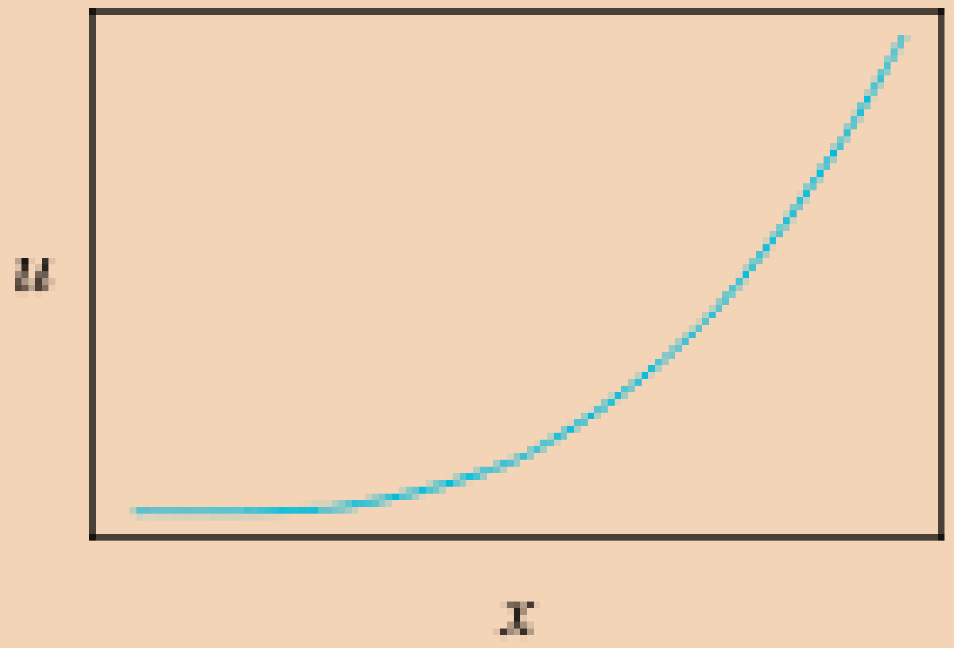
NOW SOLVE THIS

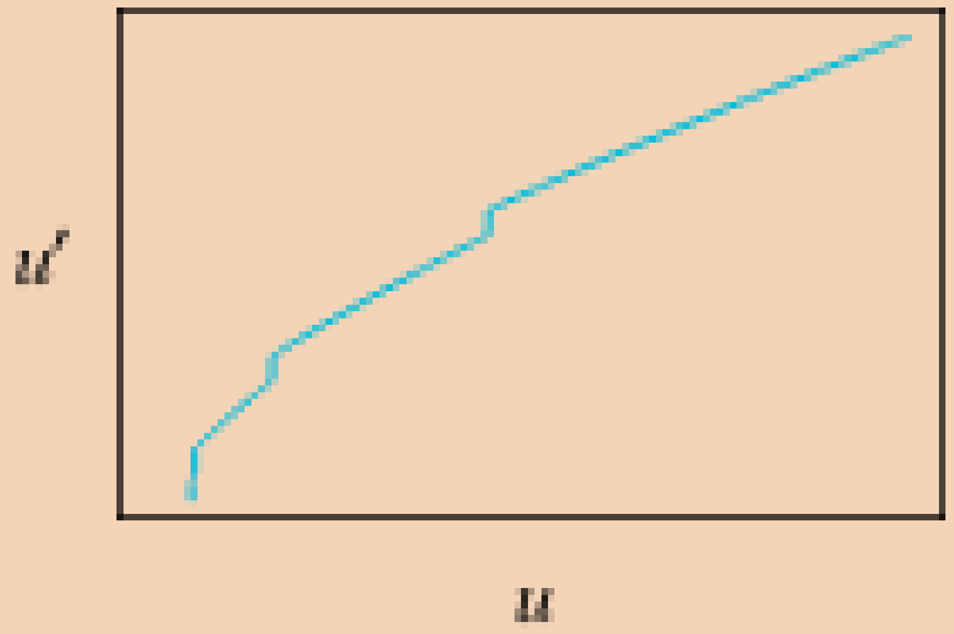
$$\frac{du}{dx} = x^2 - 1 + \frac{1}{|x^2 + 1|} + \lfloor x \rfloor$$

WHY THERE IS A PROBLEM?



CAN A COMPUTER SOLVE IT?





$u(0) = 1$

NOW YOU MIGHT BE SMART ENOUGH TO SOLVE THIS ANALYTICALLY, BUT A COMPUTER DOES WHAT YOU TELL IT TO DO.

HOW DO YOU ENCODE THIS INTO A FIXED SERIES OF STEPS?

SOME MORE BG

- PRIMITIVE APPROXIMATION METHODS RELY ON TAYLOR SERIES APPROXIMATION OF FIRST ORDER. IN THIS CASE, THE ERROR IS BOUNDED TO O(H^2).
- BUT, IN OUR CASE, THE TAYLOR SERIES DOESN'T HOLD TRUE AT PT OF DISCONTINUITY. ERROR EXPLODES.

WEAK FORMULATION

- CLASSICAL APPROACH - LOOK AT THE SPACE POINT-WISE. WE CARE ABOUT THE VALUE OF THE DERIVATIVE AT A POINT.
- WEAK FORMULATION APPROACH - LOOK AT THE AVERAGE/OVERALL BEHAVIOUR.

WEAK FORMULATION

$$u'' = f$$

$$\int_\Omega u''\phi d\Omega = \int_\Omega f\phi d\Omega = [u'\phi]_{\{\Omega\}} - \int_\Omega u'\phi' d\Omega$$

$$\int_\Omega u'\phi' d\Omega = -\int_\Omega f\phi d\Omega$$

WEAK FORMULATION

- NOW, PHI IS LIKE A PROBE FUNCTION. IT HELPS YOU ANALYZE THE FUNCTION AT A SPECIFIC POINT.
- THE COMPUTER DOES THIS FOR EVERY PT ON GRID AND THEN ASSEMBLES THE RESULT.

SOME SMALL DEFINITION BS

- L^p SPACE - THE FUNCTIONS THAT SATISFY THIS EXIST HERE.

$$||f||_p = \left(\int |f(x)|^p \right)^{1/p}$$

- THINK OF EXAMPLES OF FUNCS IN L^1 SPACE.
- WHAT ABOUT L^2 ?
- WHAT ABOUT L^∞?

SOBOLEV SPACES $W^{k,p}$

- THE SPACE IN WHICH OUR SOLUTIONS EXIST; I.E. THE WEAK FORMULATED INTEGRAL ONES.

- THE P REPRESENTS THE L^p -TH NORM.
- THE K REPRESENTS UPTO WHICH DERIVATIVE EXISTS IN THE SAME SPACE.

NOW FORGET BOTH THOSE NAMES

- WHICH PROBES DO WE USE FOR TESTING? DIRAC DELTA?

$$\int_{-\infty}^{\infty} \delta(x) = 1$$

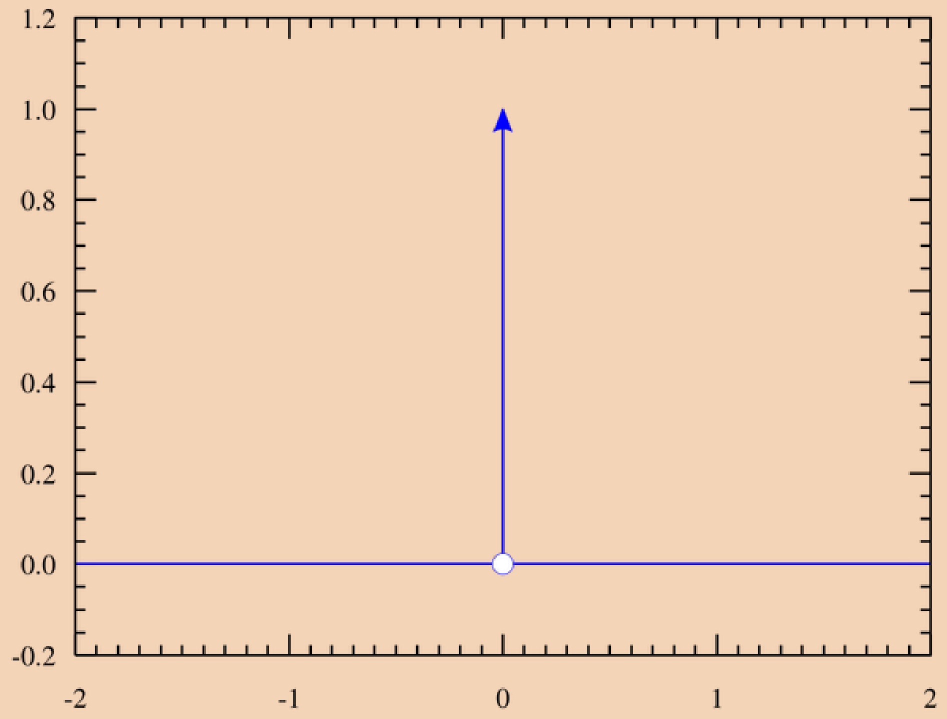
$$\int_{-\infty}^{\infty} \delta(x) f(x) = f(0), \quad \int_{-\infty}^{\infty} \delta(x-a) f(x) = f(a)$$

- NOW, WHAT IS DIRAC DELTA'S INTEGRAL OR CUMULATIVE FUNCTION? → THIS IS WHY IT'S RELEVANT TO OUR ENTIRE DISCUSSION.

A BIG NO
COZ
INFINITY ISNT REAL.

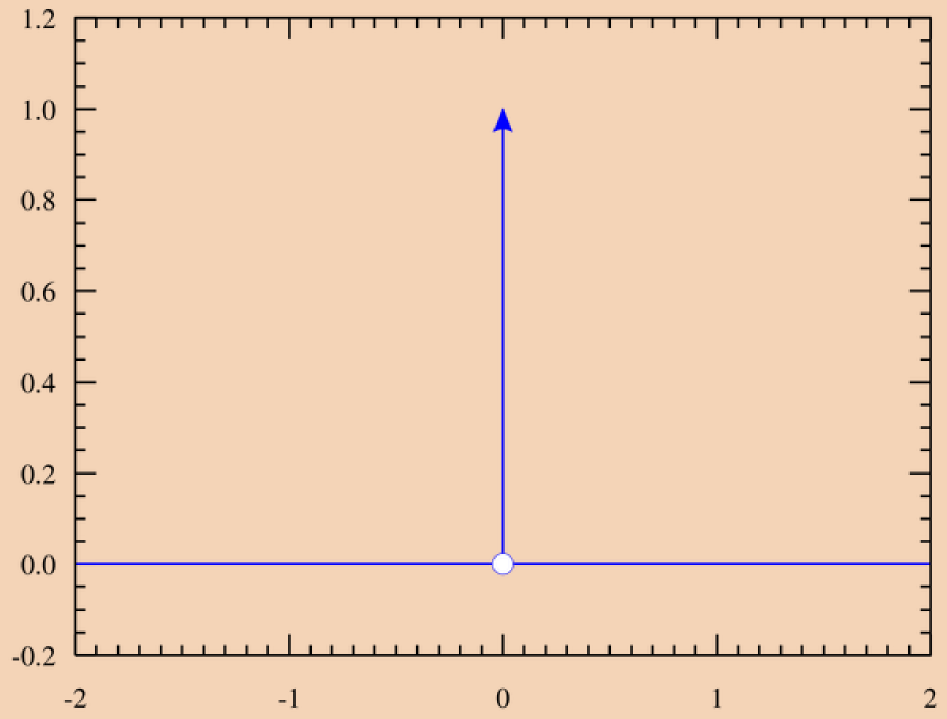
NOW FORGET BOTH THOSE NAMES

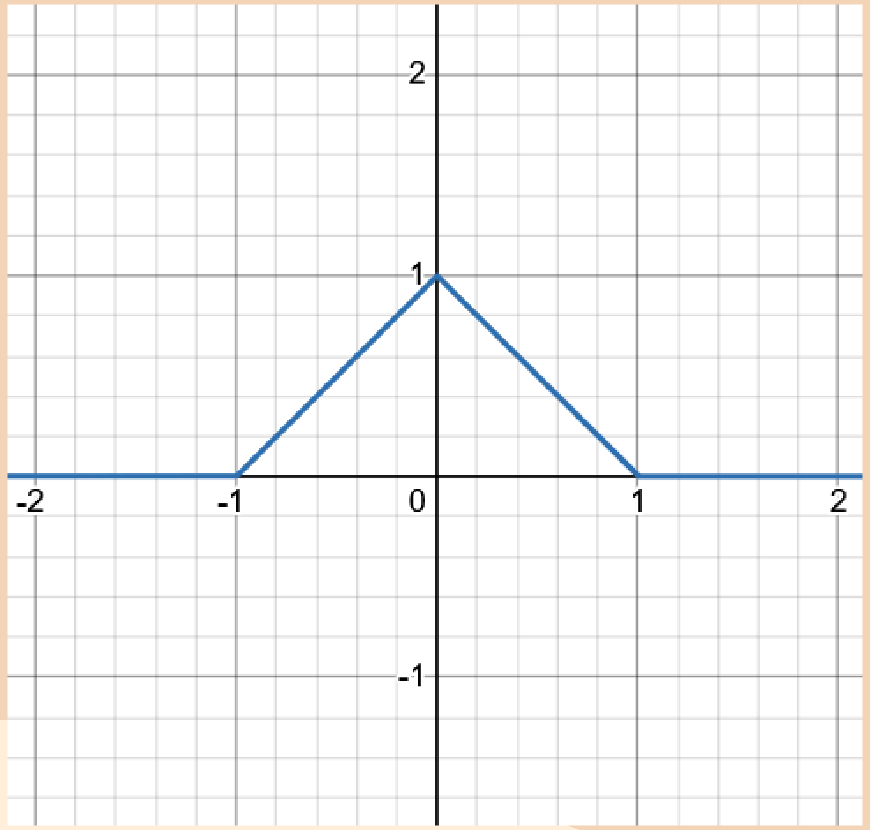
- WHAT IS THE SIMPLEST AND DUMBEST APPROXIMATION OF A DIRAC DELTA.



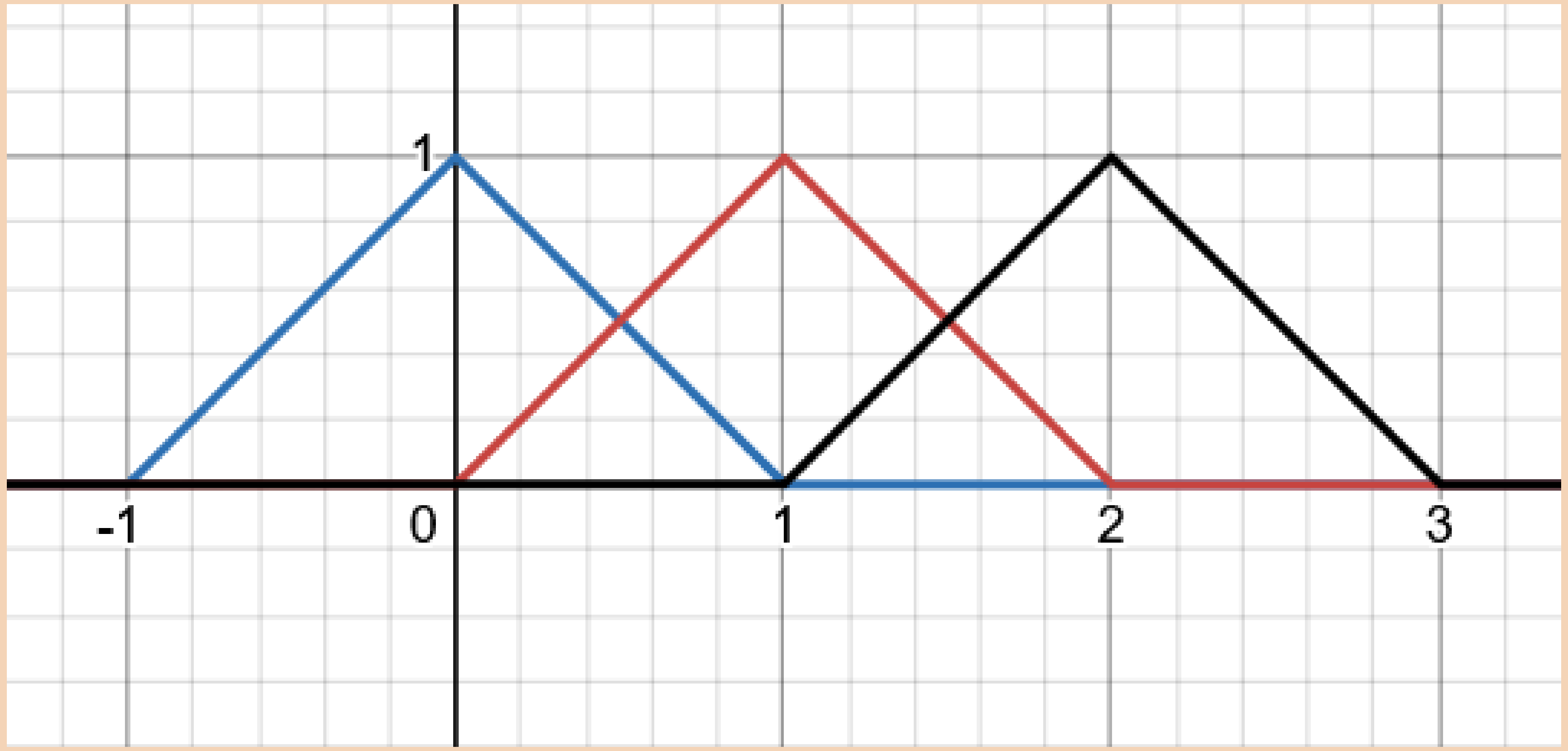
NOW FORGET BOTH THOSE NAMES

- WHAT IS THE SIMPLEST AND DUMBEST APPROXIMATION OF A DIRAC DELTA.





HOW TO USE MULTIPLE PROBES?



- CALLED “HAT” FUNCTIONS.

WHY DID YOU YAP ABOUT L^p AND $W^{k,p}$?

- FOR THE INTEGRAL $\int u'v'dx$ TO MAKE SOME SENSE
- V MUST BE L^2, SO THE "ENERGY" DOESN'T GO TO INFINITY.
- V MUST HAVE A DERIVATIVE V' THAT IS ALSO L^2.
- BY REQUIRING V BE IN A SOBOLEV SPACE (H1), WE KNOW OUR PROBES ARE "SMOOTH " TO MEASURE THE SYSTEM WITHOUT BREAKING.

ANALOGY

- THIS IS LIKE VIRTUAL WORK THEOREM GNG.
- THINK ABOUT L^2 AS ENERGY.
- $\int fv$ AS WORK DONE.

- NOW CHANGE IN ENERGY = WORK DONE. HENCE, OUR WEAK FORMULATION ACCOUNTS FOR THE VARIATION DUE TO WEIRD DISCONTINUITIES.

SMALL PEAK INTO WHAT COMPUTER DOES

2 MAIN OPTIMISATIONS.

$$u = \sum u_i \phi_i \implies \int_{-1}^{1} u' \phi_j' = \sum u_i \int_{-1}^{1} \phi_i' \phi_j'.$$

$$\int_{-1}^{1} f(x) \phi_i' \phi_j' = A_{ij} \text{ (say)}$$

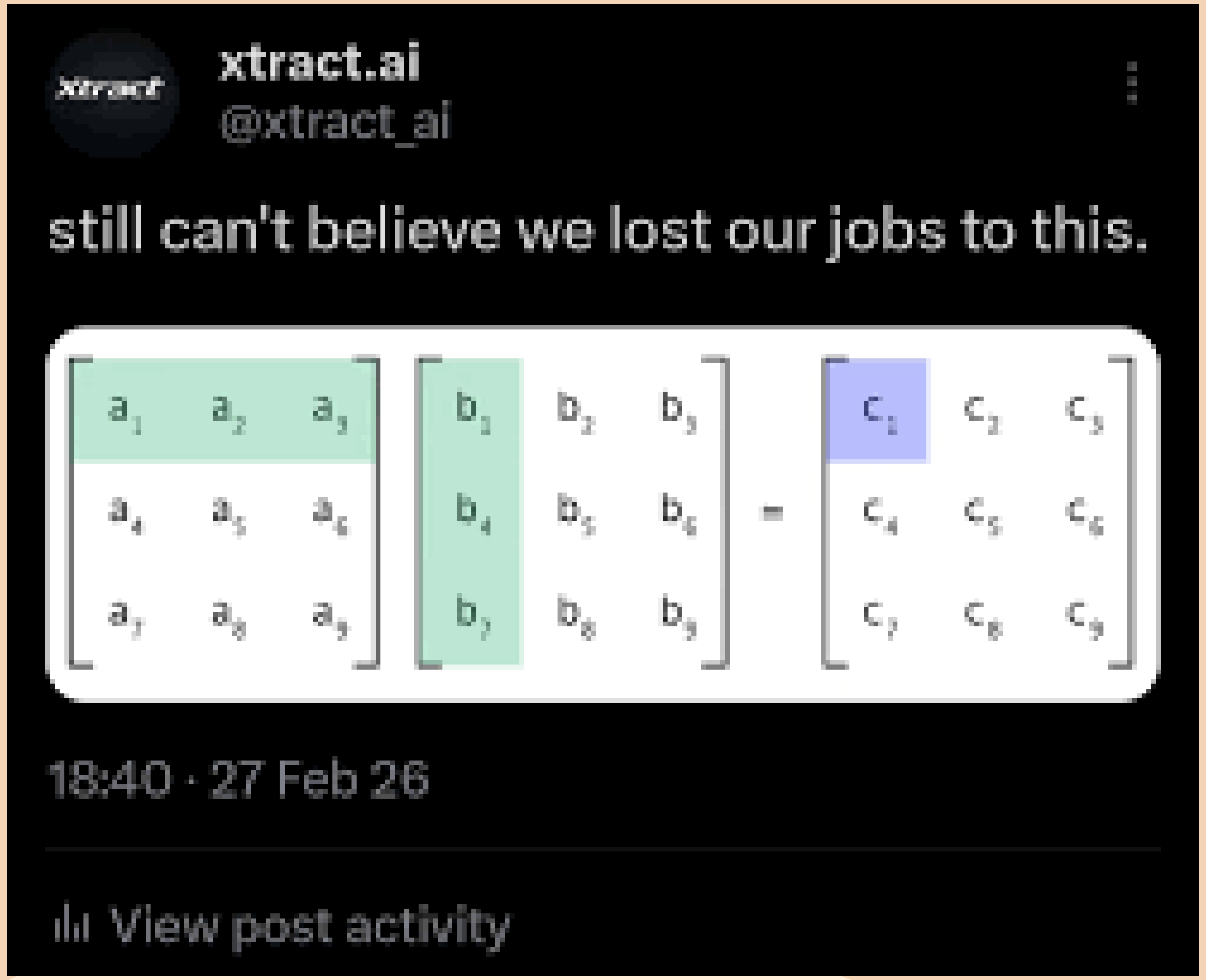
$$\int_{-1}^{1} f(x) v(x) \approx \sum_{i=1}^{N} w_k f(x_k) v(x_k)$$

→ PRE-CALCULATE.

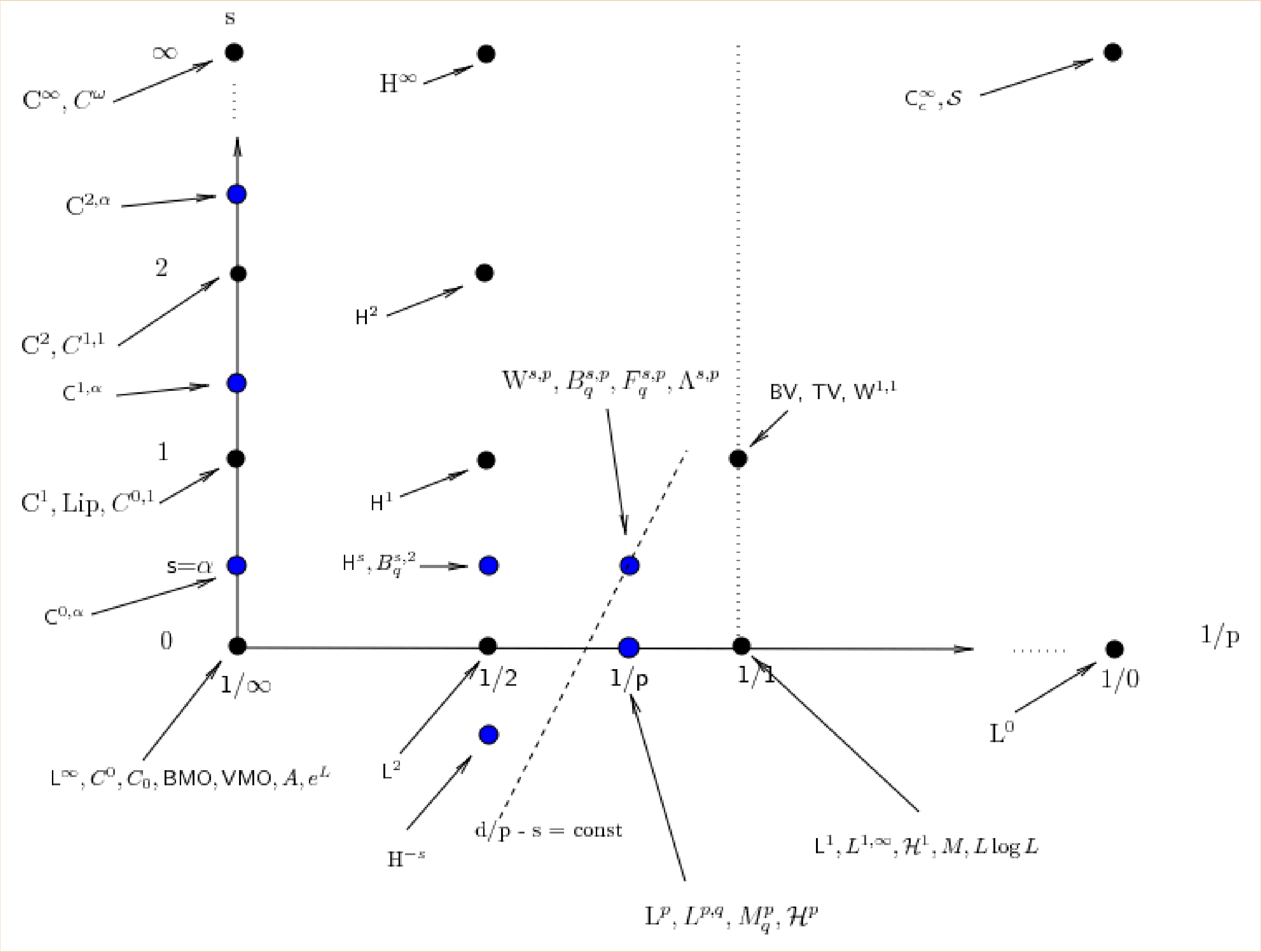
CAN'T BELIEVE WE LOST JOBS TO THIS

- THE SET OF PHI_I, PHI_J WILL RETURN A STIFFNESS MATRIX, A.
- THE VALUES U_I, U_2, ... FORM VECTOR U.
- THE INTEGRALS OF F WITH THE TEST FUNCTION FORM VECTOR B.

$$Au = b \implies u = bA^{-1}.$$



WOAH SOME STUFF TO LOOK AT



https://terrytao.wordpress.com/tag/sobolev-spaces/

HOW DOES
CALCULATIONS OF VARIATONS
FOLLOW NOW?

HOW DOES
CALCULATIONS OF VARIATONS
FOLLOW NOW?

WHAT IS ACTION?

$$J[y] = \int_{x_1}^{x_2} L\left(x, y(x), y'(x)\right) dx$$

HOW DID WE GET HERE?

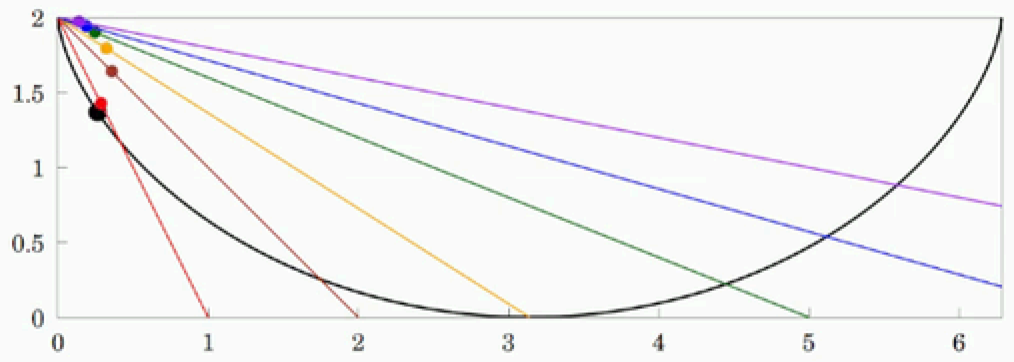
$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0$$

BELTRAMI'S IDENTITY

$$L - u' \frac{\partial L}{\partial u'} = C \, ,$$

BUT WHEN?

APPLICATIONS IN PHYSICS



Brachistochrone curve

In physics and mathematics, a brachistochrone curve, or curve of fastest descent, is the one lying on the plane between a point A and a lower point B, where B is not directly below A, on which a bead slides frictionlessly under the influence of a…

W Wikipedia

THANK YOU