Bids & Brains

PRELIMS

ANSWERS

1. $\frac{1}{2}$
2. $\ln 2$
3. $n+1$
4. 1
5. 2
6. Bonus
7. 16/3
8. 0.125
9. 1
10. 128
11. 67

SOLUTIONS

1. Swami has a regular polygon with $4n$ sides. First, he fixes a certain edge as his reference. He then chooses a random point on the circumcircle of the polygon uniformly. Now, he joins the point with the endpoints of the reference edge to form a triangle of area A. What is the expected area of A, divided by the side length and radius of the circumcircle if the point is sampled uniformly when $n \to \infty$.

Solution

For now, assume it has $n = 4k$ sides. Since we've asked the ratio of area to two quantities dependent on circumcircle radius, we can assume the radius to be 1 and proceed. Now, the area of a triangle is given by $1/2bh$, with h being the only quantity that changes here. So, $E[A] = 1/2bE[h]$. Now, let b be located parallel to $0°$. Then, the angle at which it's endpoints are located are $\theta_1 = 3\pi/2 - \pi/n$ and $\theta_2 = (3\pi/2 + \pi/n)$

$$2\pi E[h] = \int_0^{\theta_1} \sin\theta + \cos(\pi/n) + \int_{\theta_1}^{\theta_2} \cos(\pi/n) - \sin\theta + \int_{\theta_2}^{2\pi} \sin\theta + \cos(\pi/n) =$$

$$-\cos\theta_1 + 1 + \cos(\pi/n)(\theta_1) + \cos(\pi/n)(\theta_2 - \theta_1) + \cos(\theta_2) - \cos(\theta_1) - \cos(2\pi) + \cos(\theta_2) + \cos(\pi/n)(2\pi - \theta_2).$$

This humongous expression evaluates to 2π when $n \to \infty$; i.e $\theta_1, \theta_2 \to 3\pi/2$.
Applying limiting behaviour, we get $E[A]/(br) = E[h]/2 = 1/2$.

2.

$$5 \cdot \int_0^{\ln \frac{6+\sqrt{11}}{5}} \frac{\sinh x}{7 - 5\cosh x} \mathrm{d}x$$

Solution

Answer: $\boxed{\ln 2}$
Substitute $7 - 5\cosh x = u$:

$$\frac{du}{dx} = -5\sinh x$$

Therefore, the integral becomes

$$\int_2^1 5 \cdot \frac{-du}{5u} = \int_1^2 \frac{du}{u} = \ln 2$$

3. Compute the determinant of the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & \text{if } i \neq j, \\ 2, & \text{if } i = j. \end{cases}$$

Solution

Adding the second row to the first one, then adding the third row to the second one, . . ., adding the nth row to the $(n-1)$th, the determinant does not change and we have

$$\det(A) = \begin{vmatrix} 2 & -1 & +1 & \cdots & \pm1 & \mp1 \\ -1 & 2 & -1 & \cdots & \mp1 & \pm1 \\ +1 & -1 & 2 & \cdots & \pm1 & \mp1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \pm1 & \pm1 & \mp1 & \cdots & 2 & -1 \\ \mp1 & \pm1 & \pm1 & \cdots & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ \pm1 & \mp1 & \pm1 & \mp1 & \cdots & -1 & 2 \end{vmatrix}.$$

Now subtract the first column from the second, then subtract the resulting second column from the third, . . ., and at last, subtract the $(n-1)$th column from the nth column. This way we have

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n+1 \end{vmatrix} = n + 1.$$

4. Evaluate the limit

$$\lim_{n\to\infty} \left(\frac{1}{n} \cdot \sum_{k=1}^{n} k \sin\left(\frac{n}{nk}\right) \right)$$

Solution

For all $k \in \mathbb{N}$,

$$\frac{1}{k} - \frac{1}{6k^3} < \sin\left(\frac{1}{k}\right) < \frac{1}{k}$$
$$\implies 1 - \frac{1}{6k^2} < k\sin\left(\frac{1}{k}\right) < 1$$
$$\implies \sum_{k=1}^{n} \frac{1}{n} - \sum_{k=1}^{\infty} \frac{1}{6k^2 n} < \frac{1}{n} \cdot \sum_{k=1}^{n} k\sin\left(\frac{1}{k}\right) < \sum_{k=1}^{n} \frac{1}{n}$$

By sandwich theorem, the answer is $\boxed{1}$

5. Consider the functional equation:

$$f(x + f(y)) = f(x) + y \quad \forall x, y \in \mathbb{R}$$

Find the number of real valued functions, $f : \mathbb{R} \to \mathbb{R}$ that satisfy this functional equation.

Solution

Let's test for `Injectivity`:
Assume $f(a) = f(b)$: Put $y = a$ and $y = b$ in the functional equation:

$$f(x + f(a)) = f(x) + a \qquad f(x + f(b)) = f(x) + b$$

From these, we get $a = b$. This proves that the function has to be injective (one-one).
Let's test for `Surjectivity`:
Put $x = 0$:

$$f(f(y)) = f(0) + y$$

Since, RHS is unbounded as $f(0)+y \in \mathbb{R}$, we should have $f(.)$ to be able to take any real value.
This proves that the function has to be surjective(onto).
From, both of these, we can say that the function has to be `Bijective`
Now, $x \leftrightarrow y$:

$$f(y + f(x)) = f(y) + x \quad \forall x, y \in \mathbb{R}$$

Substituting this back in the given equation, we get:

$$f(f(f(x) + y)) = f(x) + y \quad \forall x, y \in \mathbb{R} \implies \boxed{f(f(t)) = t \quad \forall t \in \mathbb{R}}$$

So, $f(t) = f^{-1}(t) \quad \forall t \in \mathbb{R}$
Also, now $f(x + f(0)) = f(x) + 0 \implies f(0) = 0$
Hence, $\boxed{f(x) = x \quad \forall x \in \mathbb{R}}$ and $\boxed{f(x) = -x \quad \forall x \in \mathbb{R}}$ are the only solutions.
Answer = **2**

6. Let the points A, B, and C be collinear and the point $P \notin AB$ such that $\angle ABP = 140°$. Let O, O_0, and O_1 be the circumcenters of triangles $\triangle ABP, \triangle ACP$, and $\triangle BCP$. Find $\angle O_0 P O_1$.

Solution

Question dropped.

7. Imagine there are two cylinders of unit radius. These two cylinders intersect each other perpendicularly. Find the volume enclosed by these two cylinders.

Solution

Answer : $\boxed{\frac{16}{3}}$
We'll take 2 cylinders, one along the x-axis and the other along the y-axis.
We can find the volume of this region by finding the area along the x-y plane at some given z and then integrate that area with respect to z from -1 to 1.
At a given z, the occupied space has the shape of a square with length of the side $\sqrt{(2)^2 - (2z)^2}$.
The required area of the square is $4 - 4z^2$.
Now, to calculate the volume we'll just calculate the integral value

$$\int_{-1}^{1} (4 - 4z^2) dz = 16/3$$

Therefore, the volume enclosed is $\frac{16}{3}$.

8. There are 4 lamps, each with a switch that changes the lamp from on to off, or from off to on, each time it is pressed. The lamps are initially all off. You are going to press the switches in a series of rounds. In the first round, you are going to press exactly 1 switch; in the second round, you are going to press exactly 2 switches; and so on, so that in the 4th round you are going to press all 4 switches. In each round you will press each switch at most once.
What is the probability that after 4 rounds, all 4 lamps are on?

Solution

Answer: $\boxed{0.125}$ or $\boxed{\frac{1}{8}}$
The total number of ways in which all 4 switches are operated across the 4 rounds is:

$$\binom{4}{1} \cdot \binom{4}{2} \cdot \binom{4}{3} \cdot \binom{4}{4} = 96.$$

Of these, the favourable scenario is all 4 switches being on after the fourth round. Since all 4 switches are toggled in the fourth round, it means that all 4 switches must be off after 3 rounds. Backtracking, we get that after 2 rounds, 3 out of 4 lamps must be on; since, in the third round, all 3 lamps should be toggled off for the above scenario to occur.
Backtracking further, we get that:

- One light is switched on in the first round;
- Two of the other three lights are switched on in the second round;
- All 3 on lights are turned off in the third round.
- All 4 lights are now turned on in the fourth round.

The number of ways of all these happening are:

- First round: $\binom{4}{1} = 4$.
- Second round: $\binom{3}{2} = 3$.
- Third round: $\binom{3}{3} = 1$.

- Fourth round: $\binom{4}{4} = 1$.

Hence, the total number of ways to ensure that all 4 lights are turned on after 4 rounds is $4 \cdot 3 = 12$.
Therefore, the required probability is

$$\frac{12}{96} = \frac{1}{8} = 0.125$$

9. Find the number of functions such that $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that for all non-zero numbers x, y

$$x.f(xy) + f(-y) = x.f(x)$$
$$f(1) = 10$$

Solution

First, set $x = 1$. This gives

$$f(y) + f(-y) = f(1).$$

Next, set $y = -1$. Then we obtain

$$xf(-x) + f(1) = xf(x).$$

Using the first relation, we substitute $f(-x) = f(1) - f(x)$ into the equation above:

$$x\big(f(1) - f(x)\big) + f(1) = xf(x).$$

Solving for $f(x)$, we get

$$f(x) = \frac{f(1)(x+1)}{2x}.$$

Thus, the answer is

$$\boxed{1}.$$

10. Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many raitonal numbers betwen 0 and 1 will 20! be the resulting product?

Solution

For a fraction to be in lowest terms, its numerator and denominator must be relatively prime. Thus any prime factor that occurs in the numerator cannot occur in the denominator, and vice versa. There are eight prime factors of 20!, namely $2, 3, 5, 7, 11, 13, 17$, and 19. For each of these prime factors, one must decide only whether it occurs in the numerator or in the denominator. These eight decisions can be made in a total of $2^8 = 256$ ways.

However, not all of the 256 resulting fractions will be less than 1. They can be grouped into 128 pairs of reciprocals, each containing exactly one fraction less than 1. Thus the number of rational numbers with the desired property is $\boxed{128}$.

11. Evaluate the following expression $\sqrt[3]{21 \cdot 66 \cdot 217 + 1}$.

Solution

We can observe that:

$$21 \cdot 217 = 4557 = 67^2 + 67 + 1.$$

$$21 \cdot 66 \cdot 217 + 1 = 66(67^2 + 67 + 1) + 1 = (67 - 1)(67^2 + 67 + 1) + 1.$$

Using the identity $(x - 1)(x^2 + x + 1) + 1 = x^3$, we get:

$$\sqrt[3]{21 \cdot 66 \cdot 217 + 1} = 67.$$

Hence $\boxed{67}$ is the answer