Bids & Brains

Finals - QP - Answers

TEAM NAME:

§1 Calculus

1. **[SILVER]**

$$\lim_{n\to\infty} \sqrt{n} \int_{-\pi/2}^{\pi/2} e^{n\int_0^x \int_0^t \frac{s^2-1}{(1+s^2)^2} ds dt} dx$$

Solution

Answer: $\boxed{\sqrt{2\pi}}$
The integral in the power term evaluates to $-1/2\log(x^2+1)$.

$$\lim_{n\to\infty} \sqrt{n} \int_{-\pi/2}^{\pi/2} (1+x^2)^{-n/2} dx = \sqrt{2\pi}$$

2. **[BRONZE]** Given,

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} g(\sin^2(x))\, dx$$
$$I_2 = \int_{-\infty}^{\infty} \frac{\sin(x)}{x} g(\sin^2(x))\, dx$$
$$g(u) = \frac{1}{1+u^2}$$

Evaluate $\frac{I_1}{I_2}$

Solution

The function $g(\sin^2 x)$ is made up of powers of $\sin^2 x$ (using Taylor Series expansion). We know from trigonometry that any even power of sine, like $\sin^2 x$ or $\sin^4 x$, can be rewritten as a sum of a constant number and cosine terms like $\cos(2x), \cos(4x), \cos(6x)$, etc.
So, we can write $g(\sin^2 x)$ generally as:

$$g(\sin^2 x) = A_0 + A_1\cos(2x) + A_2\cos(4x) + \dots$$

where $A_0, A_1, \dots$ are just constants.
To solve the problem, we just need to test how the integrals handle the Constant Part (A_0) and the Cosine Parts ($\cos(2nx)$).
For I_1:

$$\int_{-\infty}^{\infty} A_0 \frac{\sin^2 x}{x^2} dx = A_0\pi$$

For I_2:

$$\int_{-\infty}^{\infty} A_0 \frac{\sin x}{x} dx = A_0\pi$$

Both integrals give the exact same value for the constant part.

For I_2 (using $2\sin A\cos B = \sin(A+B) - \sin(A-B)$):

$$\int_{-\infty}^{\infty} \cos(2kx)\frac{\sin x}{x}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(2k+1)x - \sin(2k-1)x}{x}dx$$

Since $k \geq 1$, both $(2k+1)$ and $(2k-1)$ are positive. The integrals subtract to zero:

$$= \frac{1}{2}(\pi - \pi) = 0$$

For I_1 Using $2\sin^2 x = 1 - \cos(2x)$, we expand the numerator product:

$$\cos(kx)\sin^2 x = \frac{1}{2}\cos(kx) - \frac{1}{4}\cos(k+2)x - \frac{1}{4}\cos(k-2)x$$

We apply the standard integral $\int_{-\infty}^{\infty} \frac{1-\cos(ax)}{x^2}dx = \pi|a|$ to each term (valid since coefficients sum to 0):

$$\int_{-\infty}^{\infty} \frac{\cos(kx)\sin^2 x}{x^2}dx = -\frac{\pi|k|}{2} + \frac{\pi|k+2|}{4} + \frac{\pi|k-2|}{4}$$

For any harmonic $k \geq 2$, we have $|k-2| = k-2$, so the terms cancel perfectly:

$$= \frac{\pi}{4}\left[-2k + (k+2) + (k-2)\right] = 0$$

Therefore, $\frac{I_1}{I_2} = \boxed{1}$

3. **[SILVER]** Find

$$\sum_{n=1}^{\infty} \Phi(n) \cdot \frac{1}{3^n - 1}$$

where $\Phi(n)$ is the Euler totient function defined as

$$\Phi(n) = \text{Number of positive integers} \leq n \text{ but relatively prime to } n$$

Solution

$$\sum_{n=1}^{\infty} \phi(n) \cdot \frac{1}{x^n - 1} = \sum_{n=1}^{\infty} \phi(n) \cdot \sum_{k=1}^{\infty} x^{-nk} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \phi(n)x^{-nk} = \sum_{n=1}^{\infty}\sum_{k|n} \phi(k)x^{-n}$$

$$\sum_{n=1}^{\infty}\left(\sum_{k|n} \phi(k)\right)x^{-n} = \sum_{n=1}^{\infty} nx^{-n} = \frac{x}{(x-1)^2}$$

$$\boxed{\frac{3}{4}}$$

4. **[GOLD]** Let

$$A = \int_0^1 \sin x\left[1 - (1-x)^5\right]dx$$

and

$$B = \int_0^1 \sin x\left[1 - x^5\right]dx$$

Find the value of

$$\lim_{n\to\infty} \frac{A^n}{A^n + B^n}$$

Solution

This is a very tough integral to solve analytically. Observe that we need not find the value of A and B individually to answer this. We only need to find whether $A > B$ or $A < B$.
Now, observe that $\sin x$ is increasing in [0, 1].
Also, $[1 - x^5]$ is decreasing in [0, 1]. So, $[1 - (1 - x)^5]$ is increasing in [0, 1].
From the continuous version of Chebyshev's sum inequality, we have:

- If both f and g are non-increasing or non-decreasing functions,

$$\int_0^1 f(x)g(x)dx \geq \left(\int_0^1 f(x)dx\right) \times \left(\int_0^1 g(x)dx\right)$$

- If f is non-decreasing and g is non-increasing,

$$\int_0^1 f(x)g(x)dx \leq \left(\int_0^1 f(x)dx\right) \times \left(\int_0^1 g(x)dx\right)$$

Now, consider $f(x)$ as $\sin x$ and $g(x)$ as $1 - x^5 \implies f$ is increasing and g is decreasing
So,

$$B = \int_0^1 \sin x \left[1 - x^5\right] dx \leq \left(\int_0^1 \sin x dx\right) \times \left(\int_0^1 \left[1 - x^5\right] dx\right)$$

Now, consider $f(x)$ as $\sin x$ and $g(x)$ as $1 - (1 - x)^5 \implies f$ and g both are increasing
So,

$$A = \int_0^1 \sin x \left[1 - (1 - x)^5\right] dx \geq \left(\int_0^1 \sin x dx\right) \times \left(\int_0^1 \left[1 - (1 - x)^5\right] dx\right)$$

Since $\int_0^1 \left[1 - (1 - x)^5\right] dx = \int_0^1 \left[1 - x^5\right] dx$, we have,
$\boxed{\text{A} \geq B}$
Hence, **answer = 1**

An alternate *more intuitive* Solution:

A - B $= \int_0^1 \sin x \left[x^5 - (1 - x)^5\right] dx = \int_0^{0.5} \sin x \left[x^5 - (1 - x)^5\right] dx + \int_{0.5}^1 \sin x \left[x^5 - (1 - x)^5\right] dx$
$= \int_0^{0.5} (\sin x - \sin(1 - x)) \left[x^5 - (1 - x)^5\right] dx$
Since both terms are greater than 0 at all times, $\boxed{A > B}$

5. **[GOLD]** Let $P_n(x)$ be a family of polynomials defined on $\mathbb{R}$ as

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

Let $f(x, y) : \mathbb{R} \times [-1, 1] \to \mathbb{R}$ be defined as

$$f(x, y) = \sum_{n=0}^{\infty} P_n(y) x^n$$

Find $f\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)^2$

Solution

P_n is the n^{th} Legendre Polynomial.

$$f^2(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(y)P_m(y)x^{m+n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_k(y)P_{n-k}(y)x^n$$

You can show $\sum_{k=0}^{n} P_k(\cos\theta)P_{n-k}(\cos\theta) = \sin((n+1)\theta)/\sin\theta$

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_k(y)P_{n-k}(y)x^n = \sum_{n=0}^{\infty} \frac{\sin((n+1)\theta)}{\sin\theta}x^n$$

Multiply S with $2x\cos\theta$ and use trigonometric identities to obtain the relation

$$S(1 - 2x\cos\theta + x^2) = 1$$

Plug in values to obtain $\frac{4(5+2\sqrt{2})}{17}$

Scummy Short Generating Functions Method

$$f(x,y) = \frac{1}{\sqrt{1-2xy+x^2}}$$

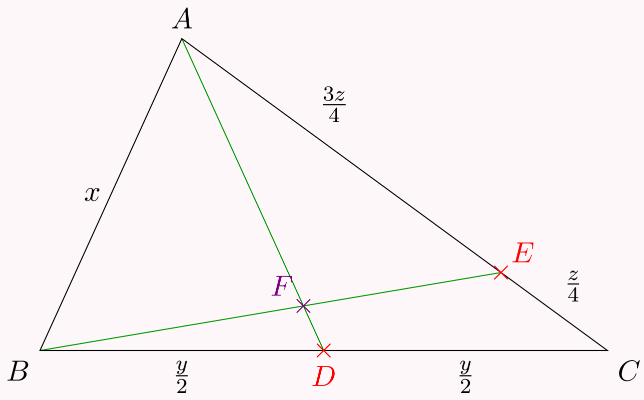
$$f\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)^2 = \frac{1}{1-\sqrt{2}^{-1}+4^{-1}} = \frac{4(5+2\sqrt{2})}{17}$$

§2 Geometry

1. [SILVER] There exists a $\triangle ABC$. D is the midpoint of edge BC and E is a point on AC such that $AE = 3\cdot EC$. F is the point of intersection of AD and BE. Find the ratio of areas of $\triangle AFB$ and $\triangle ABC$.

Solution

Answer: $\boxed{\frac{3}{7}}$ or $\boxed{0.4286}$



The above is a visual representation of the problem.
We can apply Menelaus' theorem to $\triangle ADC$ as follows:

$$\frac{AF}{FD}\cdot\frac{DB}{BC}\cdot\frac{CE}{EA} = 1.$$

We know that $\frac{DB}{BC} = \frac{1}{2}$ and $\frac{CE}{EA} = \frac{1}{3}$. Plugging in these values we get:

$$\frac{AF}{FD} \cdot \frac{1}{2} \cdot \frac{1}{3} = 1$$

$$\implies \frac{AF}{FD} = 6.$$

Now, since D is the midpoint of BC

$$Ar(\triangle ABD) = Ar(\triangle ADC) = \frac{Ar(\triangle ABC)}{2}$$

where $Ar(\triangle ABC)$ represents the area of $\triangle ABC$. The ratio of $Ar(\triangle AFB)$ and $Ar(\triangle BFD)$ is given by the ratio of their bases AF and FD respectively i.e.

$$\frac{Ar(\triangle AFB)}{Ar(\triangle BFD)} = \frac{AF}{FD} = 6.$$

Therefore, we get

$$\frac{Ar(\triangle AFB)}{Ar(\triangle ABD)} = \frac{6}{7}.$$

Hence,

$$\frac{Ar(\triangle AFB)}{Ar(\triangle ABC)} = \frac{Ar(\triangle AFB)}{2 \cdot Ar(\triangle ABD)} = \frac{3}{7}.$$

3. **[SILVER]** Given a triangle $\triangle ABC$ with base $BC = 25$ and altitude 24 dropped to that base, find the side length s of a square inscribed such that two vertices lie on BC.

Solution

Let the square be $DEFG$ with side s, where $G \in AB$ and $F \in AC$. Since $GF \parallel BC$, we have $\triangle AGF \sim \triangle ABC$ by Angle-Angle similarity.
The altitude of $\triangle AGF$ from vertex A is $h_1 = h - s$. By the properties of similar triangles, the ratio of the base to the altitude is constant:

$$\frac{s}{h-s} = \frac{b}{h}$$

Cross-multiplying yields:

$$sh = b(h-s)$$

$$sh = bh - bs$$

$$s(h+b) = bh$$

The general formula for the side length s is:

$$s = \frac{bh}{b+h}$$

$$s = \frac{25 \times 24}{25 + 24} = \frac{600}{49} = 12.244$$

4. **[GOLD]** Consider the lines

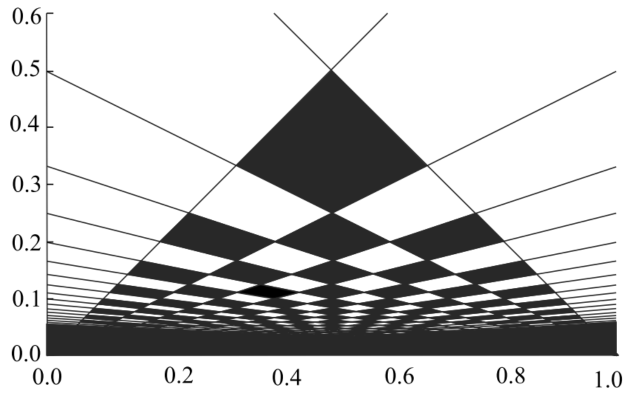
$$y = \frac{x}{1}, \quad y = \frac{x}{2}, \quad y = \frac{x}{3}, \quad y = \frac{x}{4}, \ \dots$$

and the lines

$$y = \frac{1-x}{1}, \quad y = \frac{1-x}{2}, \quad y = \frac{1-x}{3}, \quad y = \frac{1-x}{4}, \ \dots$$

restricted to the region $0 \le x \le 1$. These lines intersect to form an infinite number of quadrilaterals. Starting with the diamond-shaped quadrilateral at the top, shade every other quadrilateral, as shown in the figure.

Find the total area of all the shaded quadrilaterals.



Solution

Let $y = x/n$ be the equation of line L_n, and $y = (1-x)/k$ be the equation of line M_k, where $n, k \ge 1$. Observe that

$$L_n \cap M_k = \left(\frac{n}{n+k}, \frac{1}{n+k}\right).$$

The set of vertices of any of these tiles is

$$\{\, L_n \cap M_k,\ L_n \cap M_{k+1},\ L_{n+1} \cap M_{k+1},\ L_{n+1} \cap M_k \,\}.$$

Note that $L_{n+1} \cap M_k$ and $L_n \cap M_{k+1}$ have the same y-coordinate. Therefore, we can calculate the area of a quadrilateral to be the sum of the areas of two triangles with horizontal bases. Doing so, we find the area to be $hw/2$, where h is the difference between the y-coordinates of $L_n \cap M_k$ and $L_{n+1} \cap M_{k+1}$, and w is the difference between the x-coordinates of $L_n \cap M_{k+1}$ and $L_{n+1} \cap M_k$.
Let $A(n,k)$ denote the area of a single tile, with uppermost vertex $L_n \cap M_k$. We therefore have

$$A(n,k) = \frac{1}{2}\left(\frac{1}{n+k} - \frac{1}{n+k+2}\right)\left(\frac{n+1}{n+k+1} - \frac{n}{n+k+1}\right).$$

$$= \frac{1}{2}\left(\frac{2}{(n+k)(n+k+2)}\right)\left(\frac{1}{n+k+1}\right)$$

$$= \frac{1}{(n+k)(n+k+1)(n+k+2)}.$$

Note that each black quadrilateral has uppermost vertex $L_n \cap M_k$ with $n+k$ even. Letting $n+k = 2m$, we obtain

$$A(n,k) = \frac{1}{2m(2m+1)(2m+2)}.$$

In each horizontal row of quadrilaterals, $n+k$ is constant, and there are $n+k-1 = 2m-1$

quadrilaterals in that row. Consequently, the sum of the areas of all black quadrilaterals is

$$S = \sum_{m=1}^{\infty} \frac{2m-1}{2m(2m+1)(2m+2)}.$$

Performing partial fraction decomposition

$$S = \sum_{m=1}^{\infty} \left(\frac{-1/2}{2m} + \frac{2}{2m+1} + \frac{-3/2}{2m+2} \right).$$

$$\begin{aligned} S &= -\frac{1}{4} + \sum_{m=1}^{\infty} \left(\frac{-1/2}{2m+2} + \frac{2}{2m+1} + \frac{-3/2}{2m+2} \right). \\ &= -\frac{1}{4} + \sum_{m=1}^{\infty} \left(\frac{2}{2m+1} - \frac{2}{2m+2} \right) \\ &= -\frac{1}{4} + 2\sum_{m=0}^{\infty} \left(\frac{1}{2m+1} - \frac{1}{2m+2} \right) - 2\left(1 - \frac{1}{2}\right) \\ &= -\frac{5}{4} + 2\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \\ &= -\frac{5}{4} + 2\ln 2. \end{aligned}$$

5. **[GOLD]** Let $ABCD$ be a square sheet of paper with side length 10. The paper is folded such that the vertex C is moved to a point C' lying on the side AB. The resulting crease intersects the side BC at point P. Calculate the perimeter L of the triangle $\triangle C'BP$.

Solution

Let the side of the square be $s = 10$. Say $BC' = x$, then the perimeter of the triangle would be

$$L = BC' + BP + C'P$$

But notice that $BP + C'P = 10$ since $C'P$ is generated by the fold. Thus, $L = 10 + x$

§3 Combinatorics

1. **[SILVER]** Let N be the number of integer quadruplets (a, b, c, d) that satisfy the following system:

$$\begin{cases} a + b + c + d = 25 \\ 0 \leq a, b, c, d \leq 9 \end{cases}$$

Find the value of N.

Solution

The problem is equivalent to finding the coefficient of x^{25} in the expansion of the product of the generating functions for each variable. Since each variable a, b, c, d is constrained to $\{0, 1, \ldots, 9\}$, the generating function for a single variable is:

$$(1 + x + x^2 + \cdots + x^9) = \frac{1 - x^{10}}{1 - x}$$

Since there are 4 distinct variables, the total generating function is:

$$P(x) = \left(\frac{1-x^{10}}{1-x}\right)^4 = (1-x^{10})^4(1-x)^{-4}$$

We expand each term separately. Using the Binomial Theorem for $(1-x^{10})^4$:

$$(1-x^{10})^4 = \binom{4}{0} - \binom{4}{1}x^{10} + \binom{4}{2}x^{20} - \cdots = 1 - 4x^{10} + 6x^{20} - \ldots$$

Using the Negative Binomial Theorem for $(1-x)^{-4}$:

$$(1-x)^{-4} = \sum_{r=0}^{\infty} \binom{r+3}{3} x^r$$

We now extract the coefficient of x^{25} by multiplying relevant terms from both expansions:

a) **Term from 1:** Requires x^{25} from the second part.

$$1 \cdot \binom{25+3}{3} = \binom{28}{3} = \frac{28 \cdot 27 \cdot 26}{6} = 3276$$

b) **Term from $-4x^{10}$:** Requires x^{15} from the second part.

$$-4 \cdot \binom{15+3}{3} = -4 \cdot \binom{18}{3} = -4\left(\frac{18 \cdot 17 \cdot 16}{6}\right) = -4(816) = -3264$$

c) **Term from $+6x^{20}$:** Requires x^5 from the second part.

$$+6 \cdot \binom{5+3}{3} = +6 \cdot \binom{8}{3} = 6\left(\frac{8 \cdot 7 \cdot 6}{6}\right) = 6(56) = 336$$

Summing these values gives the final answer:

$$N = 3276 - 3264 + 336 = \binom{28}{3} - 4\binom{18}{3} + 6\binom{8}{3} = 348$$

2. **[GOLD]** Basko has constructed all binary matrices of size 8×8.A binary matrix is a matrix in which all the elements are 0 or 1.She allows you to perform the following operation. You can add any integer k to two adjacent elements such that non-negative numbers are obtained. She wants to know the number of binary matrices such that using the operation any finite number of times they can be turned into a matrix of only zeros. If the answer is $\binom{a}{b}$ such that $b = \min(a-b, b)$ give your answer as $a+b$.

Solution

Color the 8×8 board in a chessboard pattern of black and white squares. Let S_b denote the sum of the entries on black squares and S_w the sum of the entries on white squares.

In each allowed operation, the same integer k is added to two adjacent cells. Since any two adjacent cells are of opposite colors, exactly one black and one white square are affected. Therefore, both S_b and S_w change by k, and hence the difference

$$S_b - S_w$$

is invariant.

If a matrix can be transformed into the all-zero matrix, then in the final state we have

$$S_b = S_w = 0.$$

By invariance, the initial matrix must satisfy

$$S_b - S_w = 0.$$

We now show that the condition $S_b = S_w$ is also sufficient.
Consider three cells A, B, C in a row or column such that A and C are adjacent to B. Suppose their entries are a, b, c, respectively. If $a \leq b$, we may add $-a$ to both A and B, making the entry at A equal to 0. If instead $a > b$, add $a - b$ to both B and C, making the entry at B equal to a, and then add $-a$ to both A and B, making both entries 0.
Thus, by a sequence of valid operations, any positive entry can be reduced to 0 without creating negative entries.
Applying this procedure row by row, we may reduce all entries to 0 except possibly in the last two columns. Then, applying the same procedure column by column, we reduce all but two adjacent entries to 0. These final two entries must be equal because $S_b = S_w$, and hence they can also be reduced to 0.
Therefore, any matrix satisfying $S_b = S_w$ can be transformed into the zero matrix.
Now we will count the valid matrices. The board contains 32 black and 32 white squares. A binary matrix satisfies the condition $S_b = S_w$ if and only if the number of ones on black squares equals the number of ones on white squares.
For each $k \in \{0, 1, \ldots, 32\}$, there are

$$\binom{32}{k}$$

ways to place k ones on black squares and independently

$$\binom{32}{k}$$

ways to place k ones on white squares. Thus the total number of valid matrices is

$$\sum_{k=0}^{32} \binom{32}{k}^2 = \binom{64}{32}$$

Answer:

$$64 + 32 = \boxed{96}$$

3. **[GOLD]** Let S_n be the set of all permutations σ of the set $\{1, 2, \ldots, n\}$. A permutation is called an *alternating permutation* (or "zig-zag" permutation) if it satisfies the condition:

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$$

Let A_n be the number of such alternating permutations of length n. Find the value of A_7.

Solution

Let $E(n, k)$ be the number of alternating permutations of $\{1, \ldots, n\}$ ending with the number k. The recurrence relation is given by:

$$E(n, k) = E(n, k-1) + E(n-1, n-k)$$

We construct the triangle where each row sum gives A_n:

$$\begin{array}{lll} n=1: & 1 & (\text{Sum}=1) \\ n=2: & 1 & (\text{Sum}=1) \\ n=3: & 1,1 & (\text{Sum}=2) \\ n=4: & 1,2,2 & (\text{Sum}=5) \\ n=5: & 2,4,5,5 & (\text{Sum}=16) \\ n=6: & 5,10,14,16,16 & (\text{Sum}=61) \\ n=7: & 16,32,46,56,61,61 & (\text{Sum}=272) \end{array}$$

The calculation holds: $A_7 = 272$.

Alternative Approach (For the folks who know the Euler Zigzag numbers): The sequence A_n corresponds to the **Tangent Numbers** (for odd indices) and **Secant Numbers** (for even indices). The exponential generating function for A_n is given by:

$$\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = \tan(x) + \sec(x)$$

Since we require A_7 (an odd index), we examine the Maclaurin series expansion for $\tan(x)$:

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$$

We rewrite the coefficients in the factorial form $\frac{A_n}{n!}$:

$$\tan(x) = 1 \cdot \frac{x}{1!} + 2 \cdot \frac{x^3}{3!} + 16 \cdot \frac{x^5}{5!} + 272 \cdot \frac{x^7}{7!} + \dots$$

Comparing the coefficient of x^7:

$$\frac{A_7}{7!} = \frac{17}{315} \implies A_7 = \frac{17 \cdot 7!}{315} = \frac{17 \cdot 5040}{315}$$

$$A_7 = 17 \cdot 16 = 272$$

Answer: 272

4. **[SILVER]** Another crime has been commited and detective Basko has been put to the challenge of finding the culprit again. She has a total of 3114 people whom she suspects. Among the suspects there is exactly one culprit and one witness who knows whom the culprit is. To find out the culprit Basko can conduct meets with the suspects but only with exactly half the suspects at a time and then meets the other remaining half(1557 people every time). The witness will only reveal to her the name of a suspect if and only if the culprit is not present in the same meet. What is the minimum number of meets Basko must conduct. Assume that one set of two half meetings counts as two distinct meetings.

Solution

We first observe that the problem is equivalent to determining the minimum number of halvings of a set of n people such that every unordered pair of people appears in opposite halves at least once.

We claim that the minimum number of meetings required is $2k$, where k is the smallest integer such that

$$2^k \geq n.$$

Lower bound. Consider, for each person, the set of people who have never been placed in the opposite half from them. Initially, this set has size $n-1$. In any single halving, the size of such a set can decrease by at most a factor of 2, since at most half of the remaining people can be separated from a given person in one split. Therefore, after k halvings, the size of this set is at least

$$\frac{n-1}{2^k}.$$

To ensure that every such set becomes empty, we must have $2^k \geq n$. Since each halving corresponds to two meetings, this establishes a lower bound of $2k$ meetings.

Upper bound. We now show that this bound can be achieved. Label the people arbitrarily as $1, 2, \ldots, n$. We proceed by induction on n. The claim is trivial for $n = 1$.

Assume the claim holds for all integers less than n. In the first step, split the set into two halves:

$$\{1, 2, \ldots, \lfloor n/2 \rfloor\} \quad \text{and} \quad \{\lfloor n/2 \rfloor + 1, \ldots, n\}.$$

This halving separates every pair consisting of one person from each half.

Next, apply the inductive hypothesis independently to each half, recursively performing the necessary halvings within each group. At each stage, corresponding subgroups from the two halves can be arbitrarily paired to form valid halvings of the entire set.

Thus, if $T(n)$ denotes the minimum number of halvings required for n people, we have

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1.$$

Solving this recurrence yields $T(n) = k$, where k is the smallest integer such that $2^k \geq n$.

Since each halving consists of two meetings, the total number of meetings required is

$$\boxed{2k}.\text{(24 here)}$$

5. **[BRONZE]** There are 100 people waiting in a line to buy ticket to a movie. 50 among them have only a 50 Rupees note, while the other 50 have only a 100 Rupees note. The ticket seller has no change to start with. The ticket costs 50 Rupees. What is the probability that everyone will be able to buy exactly one ticket given they are not allowed to swap their places.

Solution

Answer: $\boxed{\frac{1}{51}}$

The best way to solve this question is to model it into a Dyck path and use the Catalan's Numbers.

We assign $+1$ to people with 50 note and -1 to people with 100 note. We'll use (a, b) to represent the position. In the above representation, a denotes the number of steps and b shows the final position after a steps.

We consider a random walk that starts at $(0, 0)$ and ends up at $(100, 0)$ with b never going below 0 in this process. The total number of ways in which this can be done is the Catalan number which is

$$\frac{1}{51}\binom{100}{50}$$

The total number of arrangements is $\binom{100}{50}$ which is nothing but choosing 50 random positions among the 100 positions where we can place a person who is holding a 50 Rupee Note.

Therefore the required probability is

$$\frac{\frac{1}{51}\binom{100}{50}}{\binom{100}{50}} = \frac{1}{51}$$

§4 Algebra

1. **[GOLD]** For each integer $k \geq 1$, let $S_1(k)$ denote the number of digits in the base 2 representation of 10^k, and let $S_2(k)$ denote the number of digits in the base 5 representation of 10^k. Define

$$S_1 = \{S_1(k) : k \geq 1\}, \qquad S_2 = \{S_2(k) : k \geq 1\}.$$

Let A be the number of integers in $[100, 899]$ belonging to $S_1 \cap S_2$, B be the number of integers in the range belonging to neither S_1 nor S_2 and C be the number of integers belonging to exactly one of S_1 and S_2.

Let D be defined as follows: if $2023 \in S_1 \cup S_2$, then D is the largest index k such that $S_1(k) = 2023$ or $S_2(k) = 2023$; otherwise, $D = 0$.

Find the value of $(C - A - B)(D - 4)$.

Solution

We denote the first sequence by a_k and the second sequence by b_k. Then for each positive integer k,

$$a_k = \lfloor \log_2(10^k) \rfloor + 1 = \lfloor k \log_2 10 \rfloor + 1,$$

and

$$b_k = \lfloor \log_5(10^k) \rfloor + 1 = \lfloor k \log_5 10 \rfloor + 1.$$

Let $c = \log_2 10$ and $d = \log_5 10$. Then c and d are irrational, with

$$\frac{1}{c} + \frac{1}{d} = \log_{10} 2 + \log_{10} 5 = 1.$$

Given an integer $n \geq 2$, we have

$$a_k = n \iff n < kc + 1 < n + 1 \iff \frac{n-1}{c} < k < \frac{n}{c},$$

and

$$b_k = n \iff n < kd + 1 < n + 1 \iff \frac{n-1}{d} < k < \frac{n}{d},$$

since each of $(n-1)/c$, n/c, $(n-1)/d$, and n/d is irrational.

(i) If there are positive integers i and j with $n = a_i = b_j$, then

$$\frac{n-1}{c} < i < \frac{n}{c} \quad \text{and} \quad \frac{n-1}{d} < j < \frac{n}{d}.$$

Adding the inequalities yields the contradiction

$$n - 1 < i + j < n.$$

(ii) If neither sequence contains n, then there are positive integers p and q with

$$p - 1 < \frac{n-1}{c} < \frac{n}{c} < p \quad \text{and} \quad q - 1 < \frac{n-1}{d} < \frac{n}{d} < q.$$

Once again, adding the inequalities produces a contradiction, namely

$$p + q - 2 < n - 1 < n < p + q.$$

Hence n would always be contained in either of the 2 sequences. This implies, $A = 0$, $B = 0$ and $C = 800$

(iii) For $n = 2023$, we have

$$\frac{2022}{c} \approx 608.683, \qquad \frac{2023}{c} \approx 608.984,$$

while

$$\frac{2022}{d} \approx 1413.317, \qquad \frac{2023}{d} \approx 1414.016.$$

Thus $b_{1414} = 2023$. $D = 1414$ Hence answer is $\boxed{1128000}$

2. **[BRONZE]** Let x, y, and z be complex numbers such that

$$x + y + z = 2, \qquad x^2 + y^2 + z^2 = 3, \qquad xyz = 4.$$

Evaluate

$$\frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}.$$

Express the value as $\frac{a}{b}$, where $|a|$ and $|b|$ are coprime give $a + b$ as answer.

Solution

$$xy + z - 1 = xy + 1 - x - y = (x - 1)(y - 1),$$

Likewise,

$$yz + x - 1 = (y - 1)(z - 1), \qquad zx + y - 1 = (z - 1)(x - 1).$$

Hence,

$$S = \frac{1}{(x - 1)(y - 1)} + \frac{1}{(y - 1)(z - 1)} + \frac{1}{(z - 1)(x - 1)}.$$

Combining the fractions gives

$$S = \frac{(x + y + z) - 3}{(x - 1)(y - 1)(z - 1)}.$$

Since $x + y + z = 2$, this simplifies to

$$S = \frac{-1}{(x - 1)(y - 1)(z - 1)}.$$

Expanding the denominator,

$$(x - 1)(y - 1)(z - 1) = xyz - (xy + yz + zx) + x + y + z - 1.$$

Now

$$2(xy + yz + zx) = (x + y + z)^2 - (x^2 + y^2 + z^2) = 4 - 3 = 1,$$

so

$$xy + yz + zx = \frac{1}{2}.$$

Therefore,

$$S = \frac{-1}{5 - \frac{1}{2}} = -\frac{2}{9}.$$

$\boxed{7}$ (± 7 accepted as answer)

3. **[SILVER]** Find number of functions f, defined on the set of ordered pairs of positive integers, satisfying the following properties:

   a) $f(x, x) = x$,

b) $f(x,y) = f(y,x)$,

c) $(x+y)f(x,y) = yf(x,x+y)$.

Solution

Let $f(x,y) = \operatorname{lcm}(x,y)$
It is clear that

$$\operatorname{lcm}(x,x) = x \quad \text{and} \quad \operatorname{lcm}(x,y) = \operatorname{lcm}(y,x).$$

Note that

$$\operatorname{lcm}(x,y) = \frac{xy}{\gcd(x,y)}$$

and

$$\gcd(x,y) = \gcd(x,x+y),$$

Then

$$\begin{aligned}(x+y)\operatorname{lcm}(x,y) &= (x+y)\cdot\frac{xy}{\gcd(x,y)}\\ &= y\cdot\frac{x(x+y)}{\gcd(x,x+y)}\\ &= y\operatorname{lcm}(x,x+y),\end{aligned}$$

which verifies that $f(x,y) = \operatorname{lcm}(x,y)$ satisfies the functional equation.

Now we prove that this is the only function satisfying the given conditions.
Assume, for the sake of contradiction, that there exists another function $g(x,y)$ satisfying the same conditions. Let S be the set of all pairs of positive integers (x,y) such that

$$f(x,y) \neq g(x,y),$$

and let (m,n) be such a pair with minimal sum $m+n$.
It is clear that $m \neq n$, since

$$f(m,m) = m = g(m,m).$$

By symmetry, we may assume $n - m > 0$.
Note that

$$\begin{aligned}nf(m,n-m) &= [m+(n-m)]f(m,n-m)\\ &= (n-m)f(m,m+(n-m))\\ &= (n-m)f(m,n),\end{aligned}$$

so

$$f(m,n-m) = \frac{n-m}{n}f(m,n).$$

Likewise,

$$g(m,n-m) = \frac{n-m}{n}g(m,n).$$

Since $f(m,n) \neq g(m,n)$, it follows that

$$f(m,n-m) \neq g(m,n-m),$$

so $(m,n-m) \in S$.
But $(m,n-m)$ has smaller sum

$$m + (n-m) = n < m+n,$$

contradicting the minimality of (m, n).
Therefore, our assumption is false, and $f(x, y) = \text{lcm}(x, y)$ is the unique solution. $\boxed{1}$

4. **[GOLD]** Let $x_1, \ldots, x_{90} \geq -1$ and $\sum_{i=1}^{90} x_i^3 = 0$. Find the maximum possible value of $\sum_{i=1}^{90} x_i$.

Solution

The inequality

$$0 \leq x^3 - \frac{3}{4}x + \frac{1}{4} = (x+1)\left(x - \frac{1}{2}\right)^2$$

holds for $x \geq -1$.
Let $n = 90$. Substituting $x_1, \ldots, x_n$, and summing the inequalities we obtain

$$0 \leq \sum_{i=1}^{n} \left(x_i^3 - \frac{3}{4}x_i + \frac{1}{4}\right) = \sum_{i=1}^{n} x_i^3 - \frac{3}{4}\sum_{i=1}^{n} x_i + \frac{n}{4} = 0 - \frac{3}{4}\sum_{i=1}^{n} x_i + \frac{n}{4},$$

so

$$\sum_{i=1}^{n} x_i \leq \frac{n}{3}.$$

Remark. Equality holds only in the case when $n = 9k$, k of the $x_1, \ldots, x_n$ are -1, and $8k$ of them are $\frac{1}{2}$.

5. **[SILVER]** Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that

$$P(2) = 26.$$

Solution

Let $S = \{0, 1, 2, 3\}$, and let

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0,$$

where $a_i \in S$. Then

$$P(2) = 2^m a_m + 2^{m-1}a_{m-1} + \cdots + 2a_1 + a_0.$$

We are trying to find the number of sequences $(a_0, a_1, \ldots)$ with each $a_i \in S$ such that

$$a_0 + 2a_1 + 4a_2 + \cdots = \sum_{i=0}^{\infty} 2^i a_i = n.$$

We consider the generating function

$$f(x) = (1 + x + x^2 + x^3)(1 + x^2 + x^4 + x^6)(1 + x^4 + x^8 + x^{12})\cdots,$$

where $1 + x + x^2 + x^3$ represents the different choices for a_0, $1 + x^2 + x^4 + x^6$ represents the different choices for a_1, $1 + x^4 + x^8 + x^{12}$ represents the different choices for a_2, and so on. It suffices to find the coefficient of x^n in $f(x)$.
Note that

$$f(x) = \frac{x^4 - 1}{x - 1} \cdot \frac{x^8 - 1}{x^2 - 1} \cdot \frac{x^{16} - 1}{x^4 - 1} \cdot \frac{x^{64} - 1}{x^8 - 1} \cdots = \frac{1}{(x-1)(x^2-1)}.$$

By partial fractions, we obtain

$$f(x) = \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} = \frac{-2}{4(x^2-1)} + \frac{1}{2(x-1)^2} = \frac{1}{2}\left(\frac{1}{(x-1)^2} + \frac{1}{1-x^2}\right).$$

Expanding the two functions in the last equation, we find that

$$f(x) = \frac{1}{2}\left[\left(1 - \binom{-2}{1}x + \binom{-2}{2}x^2 - \cdots\right) + (1 + x^2 + x^4 + \cdots)\right].$$

Since

$$\binom{-2}{n} = \frac{(-2)(-3)\cdots(-2-n+1)}{n!} = (-1)^n(n+1),$$

we obtain

$$f(x) = \frac{1}{2}\left[(1 + 2x + 3x^2 + \cdots) + (1 + x^2 + x^4 + \cdots)\right] = 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \cdots$$

$$= \sum_{m=0}^{\infty}\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)x^m.$$

Thus, the coefficient of x^n is $\left\lfloor \frac{n}{2} \right\rfloor + 1$, that is, there are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ polynomials satisfying the conditions of the problem.

$\boxed{14}$

§5 Miscellaneous

1. **[BRONZE]** Find the greatest common divisor of the numbers 123456789 and 987654321.

Solution

Let $T_n = 123\ldots n$ and $R_n = n(n-1)\ldots 1$. To find $\gcd(T_9, R_9)$, we first recall that

$$\gcd(T_9, R_9) = \gcd(T_9, R_9, T_9 + R_9).$$

Now,

$$T_9 + R_9 = 111\ldots 10,$$

so

$$\gcd(T_9, R_9) = \gcd(T_9, R_9, 111\ldots 10).$$

Since 10 and its factors cannot contribute to the gcd, this is equivalent to

$$\gcd(T_9 - 111\ldots 1,\ R_9 - 111\ldots 1).$$

But

$$T_9 - 111\ldots 1 = T_8 \quad \text{and} \quad R_9 - 111\ldots 1 = R_8,$$

so the problem reduces to finding $\gcd(T_8, R_8)$.
Repeating the same argument, we eventually obtain a common factor of

$$111\ldots 11 \times 9.$$

At this stage, we observe that 9 divides both numbers. Removing this common factor and continuing the process leads to

$$\gcd(T_7, R_7),$$

and repeating this reduction further shows that no additional common factors appear. Hence, $\boxed{9}$ is the only common divisor.

2. **[GOLD]** Let $(a_n)_{n=0}^{\infty}$ be a sequence with

$$\frac{1}{2} < a_n < 1 \quad \text{for all } n \geq 0.$$

Define the sequence $(x_n)_{n=0}^{\infty}$ by

$$x_0 = a_0, \qquad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} \quad (n \geq 0).$$

Find $\lim_{n\to\infty} x_n$? (answer -1, if limit does not exist)

Solution

We prove by induction that

$$0 < 1 - x_n < \frac{1}{2^{n+1}}.$$

Then we will have $(1 - x_n) \to 0$ and therefore $x_n \to 1$.
The case $n = 0$ is true since

$$\frac{1}{2} < x_0 = a_0 < 1.$$

Supposing that the induction hypothesis holds for n, from the recurrence relation we get

$$1 - x_{n+1} = 1 - \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} = \frac{1 - a_{n+1}}{1 + a_{n+1}x_n}(1 - x_n).$$

By

$$0 < \frac{1 - a_{n+1}}{1 + a_{n+1}x_n} < \frac{1 - \frac{1}{2}}{1 + 0} = \frac{1}{2},$$

we obtain

$$0 < 1 - x_{n+1} < \frac{1}{2}(1 - x_n) < \frac{1}{2} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}.$$

Hence, the sequence converges in all cases and

$$x_n \to 1.$$

3. **[SILVER]** Arjun and Lalith play a game on an 8×8 chessboard.
   - They take turns, placing a king on the board on each turn.
   - A king can only be placed on a square if it does not attack any previously placed kings.
   - The game ends when a player cannot make a valid move.

Determine the number of steps under optimal play to reach a conclusive result.

Solution

The 8×8 chessboard can be perfectly tiled by **16 disjoint 2×2 blocks**.

$$\frac{\text{64 squares}}{\text{4 squares per block}} = \text{16 blocks}$$

In any 2×2 block, it is possible to place at most one King.
Placing a King on any square within a 2×2 block attacks all three other squares in that block.
A King on the edge of a block attacks at most 2 squares of an adjacent block.
Since the adjacent block has 4 squares total, at least $4 - 2 = 2$ squares remain safe.
The game is equivalent to selecting blocks until all 16 are occupied. The total number of moves can be extended up to **16** but not more. Under optimal play from both sides, the steps hit 16.

4. **[GOLD]** Mad scientist Hemanth decides to verify certain hypotheses using 4 distinct drunk clones of Sudhanva.The clones are placed at the corners of a unit square.At $t = 0$, they start moving from their set positions in either directions with probability $p = \frac{1}{2}$ with a speed enough to cover the perimeter in exactly 5 seconds. Following this, they never switch directions in between unless they collide with other clones, which happens elastically. What is the probability that all clones are back to their original positions at $t = 15$ seconds?

Solution

To prove that the corners will be occupied at the end:
Since the collisions are elastic, instead of rebounding elastically,(assumed only for this proof, drunks are treated identical since we only want to prove that the corners will be occupied) we can assume the clones to pass through every other clone coming their way.Hence every 5 seconds, all the corners will be occupied.

Now, we determine the conditions under which the clones occupy their original corners. We know that the total velocity of the system is conserved as the clones undergo only elastic collision.
Hence, the sum of velocities at $t = 0$ should be the same at $t = 15$ Since the clones are equally spaced at the end, they all have the same average velocity over the time period.

$$v_{avg} = (c - a) \times \frac{0.8}{4}$$

where c is the number of clones that start in the clockwise direction and a refers to the rest of them. So, the displacement is

$$3 \times (c - a)$$

We require the displacement to be a multiple of 4.
Hence, $c = a = 2, c = 4, a = 4$ are the possible conditions.
Answer $= \frac{8}{16} = \frac{1}{2}$

5. **[SILVER]** An art gallery consists of an outer perimeter with $n_{out} = 10$ vertices and one interior courtyard with $n_{in} = 4$ vertices. Determine the minimum number of vertex guards G necessary to guarantee full coverage of the gallery's interior.

Solution

The total number of vertices in the system is $n = n_{out} + n_{in} = 10 + 4 = 14$. The number of holes is $h = 1$.
In a multiply-connected polygon, the number of triangles T required to tile the surface is given by:

$$T = n + 2h - 2 = 14$$

Hoffmann's generalization of the Art Gallery Theorem states that for a polygon with n vertices and h holes, the number of guards G required is at most:

$$G = \left\lfloor \frac{n+h}{3} \right\rfloor = \left\lfloor \frac{14+1}{3} \right\rfloor = \frac{15}{3} = 5$$

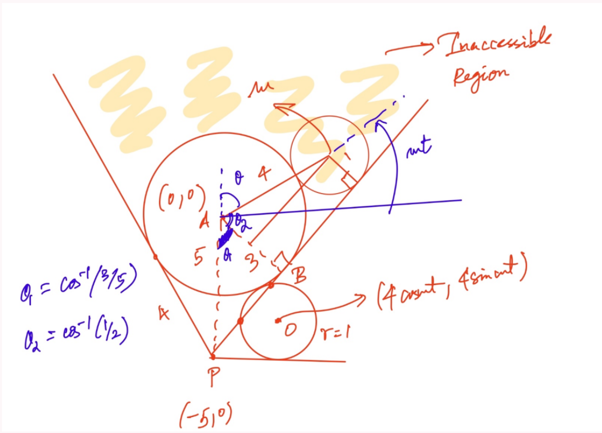
In a simple 14-vertex polygon ($h = 0$), only $\lfloor 14/3 \rfloor = 4$ guards would be required. The addition of a 4-sided hole creates an "annulus" (ring-like) topology. This geometry creates additional "pockets" where visibility can be obstructed, necessitating an additional guard (5 vs 4) to ensure that no part of the hallway remains in a shadow.
Final Answer: 5 guards are required.

§6 The one on the board (aka Q26)

[BRONZE] There is a circular fence of radius 3. Now, there's a revolving circular target (touching the fence) of radius 1 whose center rotates around the center of the fence at some speed ω. Smitali standing at a point with a distance 5 from the center of the fence. At every instant, she fires two bullets towards the target in any direction which can hit it, one long range and one short range. The long range hits the point on the circumference farther from her and the short range hits the one closer. But, SmitaliS can't shoot through the fence as it's very sturdy. Now, her score is calculated as the product of the shot distances. If her average score over one revolution of the skeet is of the form $a + b(\sin c - \cos c)$, enter $a + \pi b$.

Solution

Answer: 50.



At every instant, her score is given by $PX \cdot PY$, where X, Y are the points of intersection the bullet trajectory with the target. This quantity is also called **Power of a Point** and can be given by $PX \cdot PY = OP^2 - r^2$, where O is the center of the target, P is where Smitali stands and r the target's radius. For simplicity, let's assume the fence is at $(0, 0)$ and Smitali at $(-5, 0)$. This is evaluated to be

$$S = (4\cos(\omega t) + 5)^2 + (4\sin(\omega t))^2 - 1 = 40(1 + \cos(\omega t)).$$

Now, the accessible region for the bullets will be the region below the tangents to the fence from P. This is shown in the figure. We can figure out the angle by some trigonometry and it comes out to be

$$\theta = \arccos\left(\frac{4\sqrt{3} - 3}{10}.\right)$$

Thus, the admissible range will be $\omega t \in [0, \theta] \cup [\pi/2 + \theta, 2\pi]$. Since ω is constant, we can write the average score as $S_A = \int S(t) \cdot dt/T$, where T is time period. This comes out to be

$$S_a = \frac{1}{2\pi}\left[\int_0^\theta 40(1 + \cos(x))x + \int_{\pi/2+\theta}^{2\pi} 40(1 + \cos(x))x\right].$$

This comes out to be $30 + 20/\pi(\sin\theta - \cos\theta)$. Thus the answer is $\boxed{50.}$