Srinivasa Ramanujan Mathematics Competition 2024

Round 1 (Prelims Solutions)

Conducted on: 21st July 2024

Problem 1 Pradyumnan and Atreya play a game on a 7×7 matrix (currently empty), placing *unique* numbers from 1 to 49 (both inclusive) in each location. Pradyumnan starts, placing a number between 1 and 49 in any cell. Atreya follows, placing another number in a different cell. They alternate turns until the matrix is filled with all 49 numbers. Pradyumnan wins if the determinant of the resulting matrix is divisible by 49, and Atreya wins otherwise. Both play optimally and want to win. Who wins? Enter 1 if Atreya wins and 2 if Pradyumnan wins.

Solution Pradyumnan *always* wins if he uses the following strategy. Let the elements of the 7×7 matrix be indexed by a_{ij} where i is the row number and j is the column number of the matrix element. Pradyumnan starts by placing the number 49 at a_{7j} where j is some column number.

- Let us say that Atreya places a number k in the entry a_{ij}. If i is odd and not 7, Pradyumnan places the number $49 - k$ in the location $a_{(i+1)j}$.
- Let us say that Atreya places a number k in the entry a_{ij}. If i is even, Pradyumnan places the number $49 - k$ in the location $a_{(i-1)j}$.
- Let us say that Atreya places a number k in the entry a_{ij}. If i is 7, Pradyumnan places the number $49 - k$ in the location a_{im} where m is some column not equal to i.

Since Pradyumnan starts by placing 49 in the seventh row, there are six empty matrix elements in the last row. Every other number from 1 to 48 has a *complement* (a number which can be added to it to equal 49). Consider the first and second rows of this matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{17} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{71} & a_{72} & a_{73} & \cdots & a_{77} \end{bmatrix}$$

If Atreya places numbers in the seventh row, Pradyumnan also does and hence after 3 such pairings at some point in the game the seventh row is exhausted. No matter what Atreya plays, the elements of the first two rows added up column wise $a_{1j} + a_{2j}$ equals 49. This is true for the third and fourth rows as well as the fifth and sixth. The value of the determinant is unchanged under the simple row transformation

$$R_1 \to R_1 + R_2 \implies |A| = \begin{vmatrix} a_{11} + a_{21} & a_{12} + a_{22} & a_{13} + a_{23} & \cdots & a_{17} + a_{27} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{71} & a_{72} & a_{73} & \cdots & a_{77} \end{vmatrix}$$

$$\implies |A| = 49 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{71} & a_{72} & a_{73} & \cdots & a_{77} \end{vmatrix}$$

Therefore Pradyumnan can always ensure that the determinant of the resulting matrix A is always divisible by 49. Hence the answer is $\boxed{2}$.

Problem 2 The polynomials P(x), Q(x), R(x), S(x) satisfy the equation

$$P(x^5) + \frac{x}{3}Q(x^{10}) + \frac{x^2}{3^2}R(x^{15}) = \left(1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \frac{x^4}{3^4}\right) S(x^3) + (3x+2)^2$$

Find the value of

$$P(3^5) + Q(3^{10}) + R(3^{15})$$

Solution We are given that

$$P(x^5) + \frac{x}{3}Q\left(x^{10}\right) + \frac{x^2}{3^2}R(x^{15}) = \left(1 + \frac{x}{3} + \cdots + \frac{x^4}{3^4}\right) S(x^3) + (3x+2)^2$$

Substitute the 5^{th} roots of 3^5 (except 3 itself) in the above equation. Let us denote the above 4 numbers by $x = 3\alpha_1$, $3\alpha_2$, $3\alpha_3$ and $3\alpha_4$ where α_i is a fifth root of unity. Therefore for $i \in \{1, 2, 3, 4\}$

$$P(3^5\alpha_i^5) + \frac{3\alpha_i}{3}Q(3^{10}\alpha_i^{10}) + \frac{3^2\alpha_i^2}{3^2}R(3^{15}\alpha_i^{15}) = \left(1 + \frac{3\alpha_i}{3} + \cdots + \frac{3^4\alpha_i^4}{3^4}\right) S(3^3\alpha_i^3) + (9\alpha_i+2)^2$$

We know that $\alpha_i^5 = \alpha_i^{10} = \alpha_i^{15} = 1$ and $\sum_{k=1}^{4} \alpha_i^k = -1$. Therefore

$$\begin{aligned} & P(3^5) + \alpha_i Q(3^{10}) + \alpha_i^2 R(3^{15}) = 0 + (9\alpha_i+2)^2 \\ \implies & P(3^5) + \alpha_i Q(3^{10}) + \alpha_i^2 R(3^{15}) = 0 + 81\alpha_i^2 + 36\alpha_i + 4 \\ \implies & \alpha_i^2(R(3^{15}) - 81) + \alpha_i(Q(3^{10}) - 36) + P(3^5) - 4 = 0 \end{aligned}$$

Now we consider the below quadratic equation.

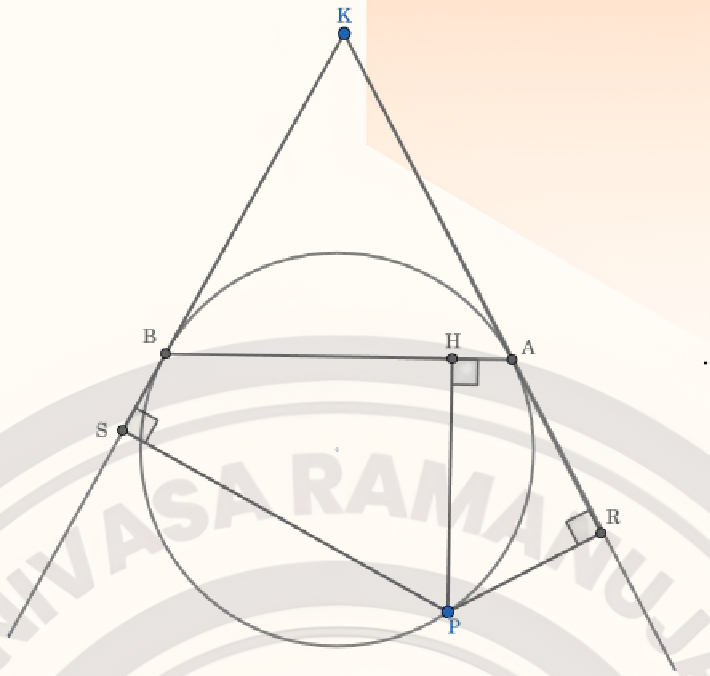
$$x^2(R(3^{15}) - 81) + x(Q(3^{10}) - 36) + (P(3^5) - 4) = 0$$

There are ***four*** roots. This implies that the equation must be identically zero. Hence $R(3^{15}) = 81$, $Q(3^{10}) = 36$ and $P(3^5) = 4$. Therefore

$$P(3^5) + Q(3^{10}) + R(3^{15}) = 81 + 36 + 4 = \boxed{121}$$

Problem 3 Let AB be a chord of a circle S. P is any point on the circle. It is given that the perpendicular distance between P and AB is 6 units. If the perpendicular distance between P and the tangent at A is 4 units, what is the perpendicular distance between P and the tangent at B?

Solution Let AB be the chord and P be any point on the circle. Let the foot of the perpendiculars on the tangents to the circle at A and B be R and S respectively, with H being the foot of the perpendicular on the chord AB.



In ΔPSB and ΔPHA, by the *Alternate Segment Theorem*, $\angle PBS = \angle PAH$. By AA similarity, these two triangles are similar. Similarly ΔPRA and ΔPHB are also similar. Therefore

$$\frac{PS}{PH} = \frac{PB}{PA}$$

$$\frac{PR}{PH} = \frac{PA}{PB}$$

Multiplying these equations, we get

$$PS \cdot PR = PH^2$$

Hence $PS = \frac{6^2}{4} = \boxed{9}$.

Problem 4 You are given two hourglasses which can measure 11 minutes and 7 minutes (the sand in each of the hourglasses takes that much time to run out). You don't have access to any other clocks and you cannot determine the amount of sand in each hourglass at an arbitrary time. You can claim that you can measure a particular time interval t minutes if and only if you can specify a 'start' and an 'end' time and the time interval in between these times equals t minutes. How many of the below time intervals can be measured using the above definition?

- 17 minutes
- 9 minutes
- 1 minute

Solution We can show that the hourglasses form a vector space.

- You can measure 11 minutes by waiting for the sand in that hourglass to run out. You can flip the hourglass over and measure another 11 minutes. Extending this argument, all time intervals which are multiples of 11 minutes can be measured. Similarly, you can measure all intervals which are multiples of 7 minutes. This denotes a meaningful notion of scalar multiplication.

- You can flip the 7-minute hourglass over after the 11-minute hourglass runs out, which means you can measure 18 minutes. This defines a meaningful notion of addition.
- We can flip both hourglasses over simultaneously and begin our 'start' time after the 7-minute hourglass runs out. We can therefore measure 4 minutes. This defines a meaningful notion of subtraction (alternatively negative scalar multiplication).

You can never measure 11 minutes using the 7-minute hourglass alone or vice versa. This implies orthogonality. Therefore the hourglasses form a vector space whose span is all nonnegative integers. This means that given a positive integer n,

$$\exists\ a, b \in \mathbb{Z} : a \times 11 + b \times 7 = n$$

Alternatively we know that any first order indeterminate equation has infinite solutions if the constants are co-prime (Aryabhata). Therefore all three time intervals can be measured. In this case it is also easy to measure the intervals.

- Flip both hourglasses simultaneously. When the 7-minute hourglass runs out, flip it. Next the 11-minute hourglass runs out. Flip it. Now the 7-minute hourglass runs out again. Flip it. Wait until the 7-minute hourglass runs out.
- Using the 11-minute hourglass we have measured 22 minutes. Using the 7-minute hourglass we have measured 21 minutes. Therefore you can 'start' when the 7-minute hourglass runs out (for the third time) and 'end' when the 11-minute hourglass runs out (for the second time).

Flipping the 11-minute hourglass 4 times and the 7-minute hourglass 6 times, we can measure 2 minutes. Proportionately scale and measure large time intervals. Therefore the answer is $\boxed{3}$.

Problem 5 Consider an infinite row of cells (empty boxes). Each cell can be in either of two states: *on* or *off*. These states can change over time (discrete steps). Consider a process called *Infection* defined below:

- If neither of the neighbour cells of a particular cell are infected, that cell recovers in the next time step (turns *off*).
- If exactly one of the neighbour cells are infected, the cell under consideration also becomes infected in the next time step (turns *on*).
- If both the neighbour cells are infected, the cell under consideration **recovers** in the next time step (turns *off*).

Initially one cell is infected (call this the *origin* cell). Call this time t = 0. Denote the cell to the immediate right of the *origin* cell to be the *friend* cell. Consider the states of the *friend* cell at time steps t = 1, 63, 185, 273, 510. How many times among the 5 time steps given is the *friend* cell infected or equivalently, *on*?

Solution Denote the state *on* by 1 and the state *off* by 0. A possible state of the infinite row of cells is given below.

1	0	1	0	1	0	1	0

Recognize that the process *Infection* is equivalent to addition modulo 2 (binary addition) through time. If we wrote down the states of all the cells one below the other, we observe that the pattern is roughly the parity of the Pascal's triangle (Piṅgala's Meru Prastara, c. 300 BC).

0	0	0	0	1	0	0	0

0	0	0	1	0	1	0	0

0	0	1	0	0	0	1	0

0	1	0	1	0	1	0	1

The pattern is the Pascal's triangle modulo 2 sampled at every even binomial exponent. Every odd binomial exponent makes no physical sense in this problem: it only serves as a mathematical abstraction of the states of the cells at *half* a time step. Therefore the *friend* cell is *on* at the k-th time step if and only if $\binom{2k}{k+1}$ is odd. This is true only if k is of the form $k = 2^m - 1$ for some m. Only $t = 1$ and $t = 63$ satisfy this form. Therefore the answer is $\boxed{2}$.

Problem 6 One day Nikhil decided that the Greatest Integer Function (hereafter referred to as GIF) is too boring. The GIF of a *real* number x is the greatest integer less than or equal to x.

- Nikhil extends this definition to the complex numbers as $[a + bi] = [a] + [b]i$.
- Karthikeya, who loves writing complex numbers in their polar form is shocked. He defines $[r \exp(i\theta)] = [r] \exp(i[\theta])$ where θ is represented in *degrees*.

A complex number is called *nice* if its Nikhil GIF and Karthikeya GIF are the same. A *nice GIF* is defined as the GIF of a nice number. For complex numbers with real and imaginary parts a, b such that $a \in [0, 13)$ and $b \in [0, 13)$ how many possible *nice GIFs* exist?

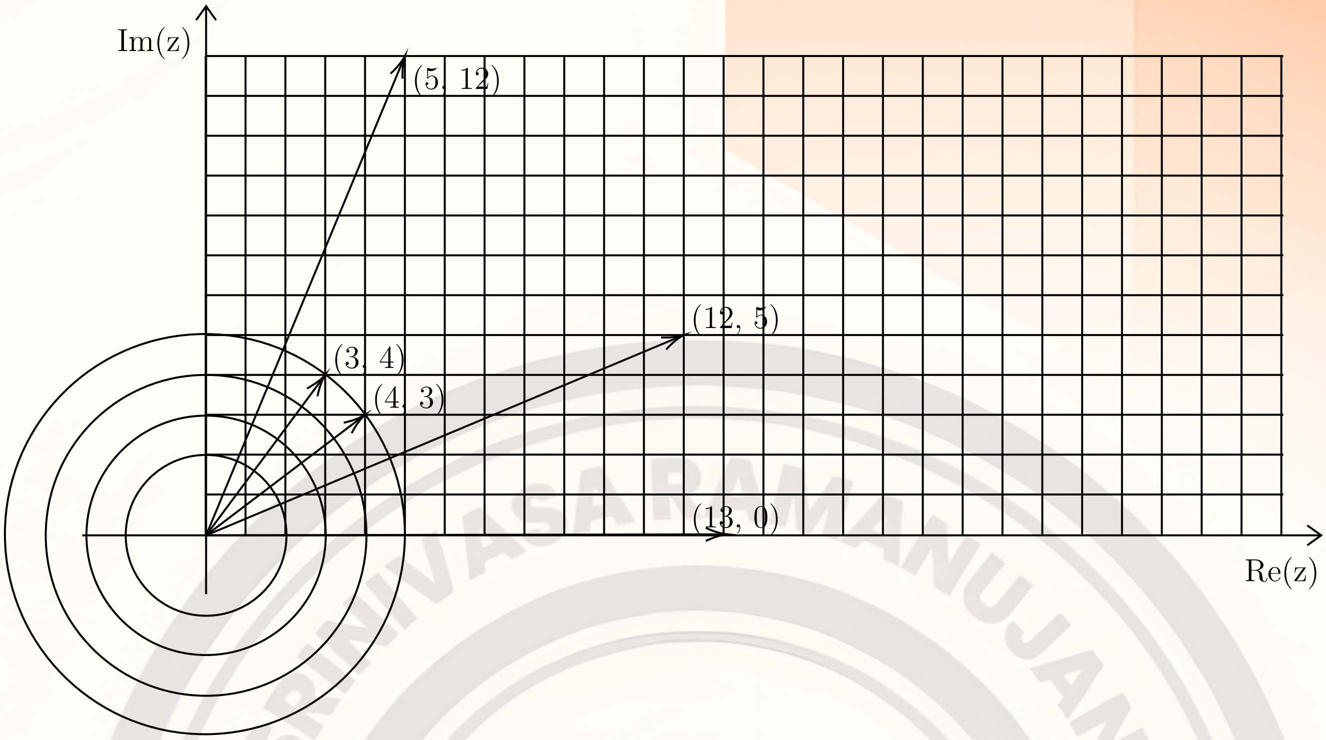
Solution We draw the Argand plane and a grid of cells of unit area on it. Each of the cell vertices are integral points (both coordinates are integers).

- The Nikhil GIF of a complex number $a + bi$ is simply the complex number corresponding to the bottom left corner of the grid cell $a + bi$ exists in.
- We then draw a series of quarter circles (we're only interested in the first quadrant). Imagine dividing these circles into arcs of 1 degrees.
- The Karthikeya GIF of a complex number falls to the immediate smaller circle and the arc with the smaller integral degree. If the GIF is *nice* then

$$[a] + [b]i = [\sqrt{a^2 + b^2}] \exp\left(i \left[\tan^{-1}\left(\frac{b}{a}\right)\right]\right)$$

$$\implies \sqrt{[a]^2 + [b]^2} = [\sqrt{a^2 + b^2}] \text{ and } \left[\tan^{-1}\left(\frac{b}{a}\right)\right] = \tan^{-1}\left(\frac{[b]}{[a]}\right)$$

Graphically this is much easier to solve. A *nice GIF* must be a grid point, lie on a circle with integral radius and must be the result of an integral degree.



Therefore we need to find the number of Pythagorean triples exist within $[0, 13)$ (in addition to this, the degree also must be an integer). Triplets like $(12, 5, 13)$ and $(3, 4, 5)$ do not have angles with integral amount of degrees. Therefore the only complex numbers which satisfy both conditions are $0 + 0i$, $1 + 0i$, $2 + 0i$, $\ldots$ $13 + 0i$, $0 + 1i$, $0 + 2i$, $\ldots$, $0 + 13i$. Hence the number of nice GIFs is $13 \times 2 - 1 = \boxed{25}$.

Problem 7 Consider the following integral-

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\cos^2(x-y) - 2\cos(x-y)\cos(x+y) + \cos^2(x+y)}{x^2y^2} dxdy$$

Find $[\sqrt{I}]$. $[a]$ is defined as the greatest integer less than or equal to a.

Solution Looking at the integral, we see that the numerator is a perfect square. Simplifying,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\cos^2(x-y) - 2\cos(x-y)\cos(x+y) + \cos^2(x+y)}{x^2y^2}\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\cos(x-y) - \cos(x+y))^2}{x^2y^2}\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{4\sin^2 x \sin^2 y}{x^2y^2}\, dx\, dy$$

One method of solving this integral is to seperate the integrals (because both functions are absolutely integrable). Alternatively, consider the 2D function $f(x, y) = \dfrac{\sin x}{x}\dfrac{\sin y}{y}$. This function can be thought of as the inverse 2D Fourier Transform of a rectangular frequency domain spread. The above integral is simply (4 times) the energy of the signal which is preserved by the Fourier Transform (Parseval's Theorem). Consider two orthogonal frequencies u and v. The non-unitary inverse Fourier Transform is given by

$$f(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v)e^{i(ux+vy)}\, du\, dv$$

$$= \frac{1}{4\pi^2}\frac{2i\sin(\frac{\omega_1 x}{2})}{ix}\frac{2i\sin(\frac{\omega_2 y}{2})}{iy}$$

$$= \frac{1}{\pi^2}\frac{\sin(\frac{\omega_1 x}{2})}{x}\frac{\sin(\frac{\omega_2 y}{2})}{y}$$

Comparing with our expression, $\omega_1 = \omega_2 = 2$. By Parseval's Theorem (for 2 dimensions)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\,dy = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)|^2\,du\,dv$$

Hence our integral is $4\pi^2$. We are asked $[2\pi] = [6.28\ldots] = \boxed{6}$.

Problem 8 Evaluate the absolute value of the sum of the components of the resultant matrix/vector

$$\int \begin{bmatrix} 3y & 3x \\ 4 & 2y \end{bmatrix} d\begin{bmatrix} x \\ y \end{bmatrix}$$

from (3, 4) to (0, 5) along the circle centered at the origin anticlockwise.

Solution Multiplying the matrices we get

$$\int \begin{bmatrix} 3y\,dx + 3x\,dy \\ 4\,dx + 2y\,dy \end{bmatrix} = \int \begin{bmatrix} 3\,d(xy) \\ 4\,dx + 2y\,dy \end{bmatrix}$$

Since this is in terms of complete differentials, the result of the integral is path independent (the fact that you traverse along the circle is irrelevant). Substituting limits after integrating, we get $[-36 \;\; -3]^T$. Hence the answer is $\boxed{39}$.

Problem 9 Consider two sets A and B defined below.

- Set A contains *all* vectors of the form $(a_1, a_2, a_3, ...)$ where $a_i \in \mathbb{N} + \{0\}$ and $\exists n \in \mathbb{N} : a_k = 0 \;\forall\; k > n$. In other words the number of choices of each element is infinite (all non-negative integers) but the number of such elements is finite.
- Set B contains *all* vectors of the form $(b_1, b_2, b_3, ...)$ where $b_i \in \{0, 1\}$. Now the number of choices for each element is finite (2) but the number of such elements is infinite.

Let $|X|$ denote the cardinality of a set X. If $|A| > |B|$, answer 0. If $|A| = |B|$ answer 1. Otherwise answer 2.

Solution Consider an element of the set A. Against each element of a vector in set A write the prime numbers in sequence. For example consider the vector (0, 1, 3, 12, 0, 0, 0 . . .). Write the prime numbers in ascending order (2, 3, 5, 7, 11, 13, . . .). Corresponding to each element, raise the prime number by that exponent. Multiply the resultant numbers. Calling the resultant number n, we get

$$n = 2^0 3^1 7^3 11^{12} 13^0 17^0 \ldots$$

Due to the Fundamental Theorem of Arithmetic, we know that every natural number has a unique prime factorization. Therefore each vector in A corresponds to a unique natural number. In other words we have defined a bijective map between elements of the set A and $\mathbb{N}$. Hence $|A| = |\mathbb{N}|$. The astute among you might point out issues with convergence. Luckily $\exists n \in \mathbb{N} : a_k = 0 \;\forall\; k > n$ so the number of nonzero exponents is finite. Convergence is guaranteed therefore.

Consider an element of the set B. For every vector in set B, define a number (written in base 2) with digits as components of its vectors in order. For example if there is a vector $(1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, \ldots.)$

$$n_2 = 0.101100011101...$$

In base 10, this corresponds to a decimal number between 0 and 1. Since we are given all possible vectors, this exhausts all decimal numbers in the interval [0, 1]. Hence we find a bijective map between set B and [0, 1]. It is a well known result that the cardinality of [0, 1] far exceeds the natural numbers (Cantor's Diagonalization argument). You can google why $\mathbb{R}$ is uncountable. Therefore the answer is $\boxed{2}$.

Problem 10 We define $T(A, B)$ to be the number of ways in which A candies can be distributed among B children when every child gets at least one candy. Find $\sum_{i=3}^{12} T(i + 1, 3)$.

Solution We use the standard 'stars and bars' method to solve this question. Using this, we get $T(A, B) = \binom{A-1}{B-1}$. Now

$$S = \sum_{i=3}^{12} T(i+1, 3) = \sum_{i=3}^{12} \binom{i}{2} = \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{12}{2}$$

$$S = \binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{12}{2} - 1$$

Now we use the property $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$. We can use it recursively (standard method) as shown below:

$$S = \binom{4}{3} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{12}{2} - 1 = \binom{5}{3} + \binom{5}{2} + \cdots + \binom{12}{2} - 1 = \binom{13}{3} - 1 = 285$$

Therefore the answer to the question is $\boxed{285}$.

Problem 11 Consider two sets. The first set (denoted by C_1) contains all points (x, y) which satisfy the equation $x^2 + \frac{y^2}{4} = k_1$. We will refer to this equation as the C_1 curve henceforth. The second set (denoted by C_2) contains all points (x, y) satisfying an equation referred to as the C_2 curve. At every intersection point of the C_1 curve and the C_2 curve, the tangents to the curves at these points have x and y intercepts at x_1, y_1, x_2, y_2 respectively. These intercepts satisfy the condition $y_1 y_2 = 4x_1 x_2$. If $(11, 21)$ and $(3, \alpha)$ both belong to C_2, what is the value of α?

Solution The C_1 curve is given by:

$$x^2 + \frac{y^2}{4} = k_1$$

The slope of the tangent to C_1 at a point (x_1, y_1) is:

$$m_1 = \frac{-y_1}{x_1}$$

Similarly, the slope of the tangent to C_2 at a point (x_2, y_2) is:

$$m_2 = \frac{-y_2}{x_2}$$

We are given that:

$$y_1y_2 = 4x_1x_2$$

Therefore

$$m_1m_2 = \frac{y_1y_2}{x_1x_2} = 4$$

Now, considering the equation of the C_1 curve:

$$x^2 + \frac{y^2}{4} = k_1$$

The derivative of this equation gives the slope of the tangent.

$$2x + \frac{2y}{4}\frac{dy}{dx} = 0 \implies m_1 = \frac{dy}{dx} = \frac{-4x}{y}$$

At the intersection points of C_1 and C_2, the slopes must satisfy

$$m_2 = \frac{4}{m_1} = \frac{-y}{x}$$

Thus, the equation of the C_2 curve can be written as:

$$xy = k_2$$

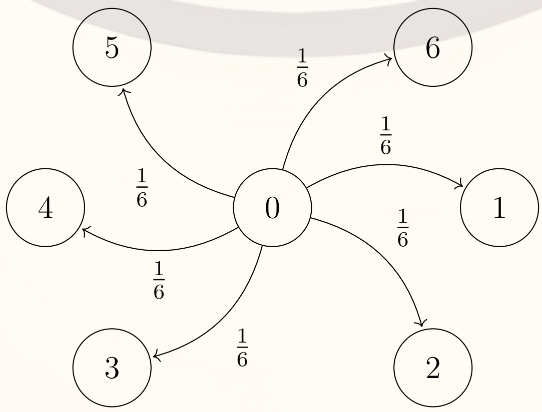
Given that $(11, 21)$ and $(3, \alpha)$ belong to C_2, we substitute these points into the equation $xy = k_2$.

$$\implies 11 \cdot 21 = 3 \cdot \alpha$$

Therefore $\alpha = \boxed{77}$.

Problem 12 What is the expected number of throws of a die until we get the total (cumulative) sum to have remainder 3 when divided by 7?

Solution Solving these questions is usually best done using Markov chains. Let $E[n]$ be expected number of throws of dice until the total is 3 mod 7 given that you started with a sum equal to n mod 7. From the definition, it is easy to see that $E[3] = 0$. From a sum of 0 mod 7, every other 6 possible states are equally likely. The below Markov diagram shows this.



$$E[0] = \frac{1}{6}(E[1]+1) + \frac{1}{6}(E[2]+1) + \frac{1}{6}(E[3]+1) + \frac{1}{6}(E[4]+1) + \frac{1}{6}(E[5]+1) + \frac{1}{6}(E[6]+1)$$

$$E[0] = 1 + \frac{1}{6}(E[1]+E[2]+E[3]+E[4]+E[5]+E[6])$$

By symmetry we can write equations like this for all other states (except for $E[3]=0$). Adding $\frac{E[i]}{6}$ to the ith equation and adding all the six equations we get

$$\implies E[0] = E[1] = E[2] = E[4] = E[5] = E[6]$$

This intuitively makes sense too. Plugging this result back into the equation we get $E[0] = E[1] = E[2] = E[4] = E[5] = E[6] = 6$. When we start the game our score is 0 mod 7, so our answer is $E[0] = \boxed{6}$.

Problem 13 If $H = \begin{bmatrix} 3 & 7 & 1 \\ -5 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{x} = [x_1, x_2, x_3]^T$ find maximum value of $\mathbf{x}^T H \mathbf{x}$ given that $||\mathbf{x}||$=1.

Solution We exploit a property of the quadratic form $\mathbf{x}^T H \mathbf{x}$ to solve this question. The maximum value of

$$f_A(\mathbf{x}) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

is equal to the largest eigenvalue of A when A is a symmetric matrix. Because $||\mathbf{x}|| = 1$, the above problem reduces to finding the maximum value of $\mathbf{x}^T H \mathbf{x}$. Therefore we need to perform eigenvalue analysis on a symmetric matrix. But H isn't a symmetric matrix! Here's a nice trick. The transpose of a scalar is itself.

$$f_H(\mathbf{x}) = \mathbf{x}^T H \mathbf{x} = (\mathbf{x}^T H \mathbf{x})^T = \mathbf{x}^T H^T \mathbf{x}$$

Therefore

$$f_H(\mathbf{x}) = \mathbf{x}^T \left(\frac{H + H^T}{2} \right) \mathbf{x}$$

Luckily $\frac{H+H^T}{2}$ is symmetric so we can apply the above property.

$$H_{\text{sym}} = \frac{H+H^T}{2} = \frac{1}{2}\left(\begin{bmatrix} 3 & 7 & 1 \\ -5 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -5 & 1 \\ 7 & 3 & 2 \\ 1 & 0 & 3 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

The eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ are $2, 2$ and 5. Therefore the answer is $\boxed{5}$.

Problem 14 You are given that f(x)=$\begin{cases} \sqrt{16-(x+8)^2} & -8 \le x \le -4 \\ x^2 - 16 & -4 \le x \le 4 \\ \sqrt{16-(x-8)^2} & 4 \le x \le 8 \end{cases}$. If the minimum value of the integral

$$\int_{-8}^{8} |f(x) - k| dx$$

over all values of k is x, find $[x]$. $[a]$ denotes the greatest integer less than or equal to a.

Solution Consider a monotonic function $f(x)$. Let us optimize

$$F(k) = \int_a^b |f(x) - k|dx$$

or equivalently

$$G(\alpha) = \int_a^b |f(x) - f(\alpha)|dx$$

where $(\alpha, f(\alpha))$ is the intersection of $y = k$ and $y = f(x)$. The same analysis applies to decreasing functions as well. For $G(\alpha)$ to be minimum clearly α should lie between a and b. Consider

$$G(\alpha) = \int_a^\alpha (f(\alpha) - f(x))dx + \int_\alpha^b (f(x) - f(\alpha))dx$$

Now

$$\frac{dG}{d\alpha} = \int_a^\alpha \frac{df(\alpha)}{d\alpha}dx + \int_\alpha^b -\frac{df(\alpha)}{d\alpha}dx = 0 \text{ (for the minimum)}$$

Therefore

$$\alpha = \frac{a+b}{2} \implies k = f\left(\frac{a+b}{2}\right)$$

In the given question, the function is even. So we can take the increasing part (for $x > 0$), integrate and then double the result to obtain the overall answer. If I is the required answer then

$$I = 2\int_0^8 |f(x) - k|dx$$

.
Since $f(x)$ is increasing in the region $x \geq 0$

$$k = f\left(\frac{0+8}{2}\right) = f(4) = 0$$

Therefore the minimum value is

$$I = 2\int_0^8 |f(x)|dx = 2\left(\int_0^4 (16 - x^2) + \text{Area of a quarter circle of radius 4}\right)$$

$$= 8\pi + \frac{256}{3} \approx 110.47$$

$$[I] = \boxed{110}$$

Problem 15 You are given an ellipse $E := \frac{x^2}{4} + \frac{y^2}{9} = 1$ with center O. The tangents of E at Q and R intersect at a point P such that $\frac{[\triangle PQR]}{[\triangle OQR]} = 8$. The locus of P is given by S. Define $F(A)$ to be the perimeter of A. Find the value of $\frac{F(S)}{F(E)}$. Note that $[\triangle ABC]$ denotes the area of the $\triangle ABC$.

(14,2.75) circle (10.5cm); [short] (13.25,25.5) – (33.75,-11.25); [short] (13.25,25.25) – (-5.25,-12.75); [short] (4.5,7) – (23.5,7); [short] (-0.25,-2.75) – (10,-7); [short] (10,-7) – (27,1); [short] (4.25,7) – (10,-7); [short] (10,-7) – (23.5,7); [short] (9.75,7) – (10,-7); [font=] at (3.5,7.5) B; [font=] at (24,8) A; [font=] at (9.5,8) H; [font=] at (-1.5,-2) S; [font=] at (27.5,1) R; [font=] at (9.5,-7.5) P;

Solution In this problem we observe the power of the 'scaling' argument. Let us scale $x \to 2x$ and $y \to 3y$. Beautifully because this is a linear transformation, the ratio $\dfrac{[\triangle PQR]}{[\triangle OQR]}$ does not change. Our ellipse becomes a simple circle in the scaled system.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \to x^2 + y^2 = 1$$

Consider the circle drawn below.

$$\frac{[\triangle PQR]}{[\triangle OQR]} = 3$$
$$\implies \frac{[OQPR]}{[OQR]} = 4$$
$$OP = 2OQ = 2$$

Therefore the locus of P in transformed axes is simply another circle with twice the radius:

$$x^2 + y^2 = 4$$

Scaling the axes back, we obtain the locus of P in our original axes: $x \to \dfrac{x}{2}$ and $y \to \dfrac{y}{3}$.

$$\frac{x^2}{4} + \frac{y^2}{9} = 4$$

The locus S has twice the semimajor and semiminor axes compared to the ellipse E. The perimeter of a shape scales linearly with the change in the semimajor and semiminor axes as long as the relative scaling remains preserved. Hence the ratio is $\boxed{2}$.

Problem 16 In $\triangle ABC, AB = 7, AC = 2$ and $II_A = 6$ where I is the incentre of the triangle and I_A is the excentre opposite to the vertex A. Find the value of $92(BC)^2$.

Solution
By the Incentre-Excentre Lemma, $BL = LC = LI = LI_A = \dfrac{z}{2}$ Now

$$\angle LBC = \angle LCB = \frac{A}{2}$$

.

$$\implies \cos\left(\frac{A}{2}\right) = \frac{a}{z}$$

We now apply the cosine rule.

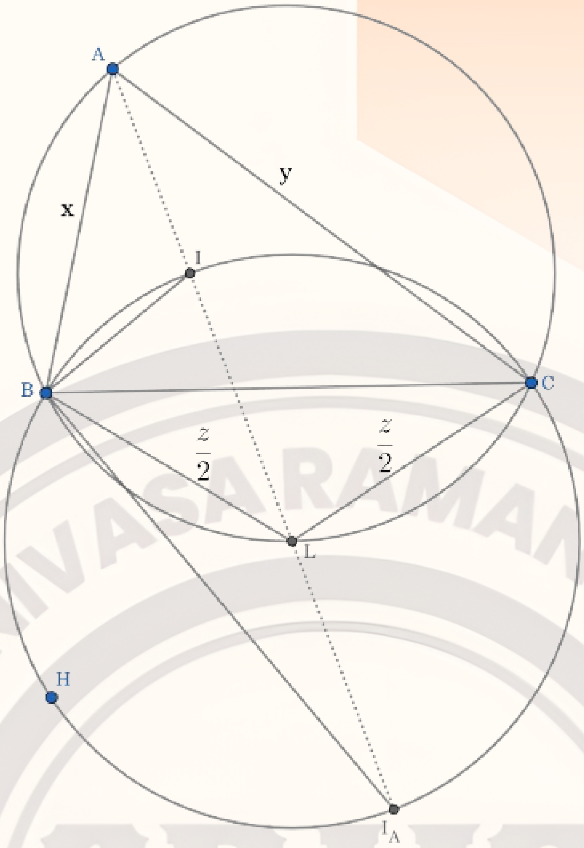
$$\cos(A) = \frac{x^2 + y^2 - a^2}{2xy} = 2\left(\frac{a}{z}\right)^2 - 1$$

Solving for a

$$a = \frac{(x+y)z}{\sqrt{z^2 + 4xy}}$$

Therefore

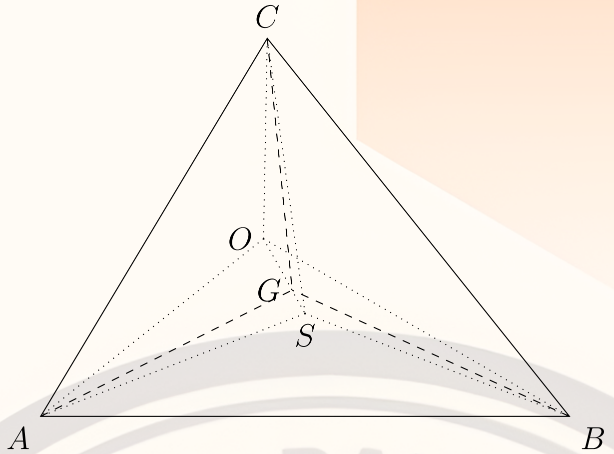
$$92(BC)^2 = 92 \times \frac{(7+2)^2 \times 6^2}{92} = \boxed{2916}$$



Problem 17 In $\triangle ABC$, $[\triangle GBS] = 5, [\triangle OCS] = 2$ then find the sum of all possible values of $[\triangle OAS]$, where G is the centroid, S is the circumcenter and O is the orthocenter. Note that $[\triangle ABC]$ denotes the area of the $\triangle ABC$.

Solution The centroid of any triangle has a few interesting properties and this problem is designed to highlight one or two of them.

1. In any triangle the orthocenter O, circumcenter S and the centroid G are collinear.
2. G divides the line joining O and S in the ratio $2:1$ internally $\Rightarrow \dfrac{OG}{GS} = \dfrac{2}{1}$
3. Sum of algebraic distances from the vertices of the triangle to any line passing through the centroid is *zero* $\Rightarrow$ Numerically (without considering direction) sum of the smaller two distances is equal to the third.
4. Triangles with same bases have their areas proportional to their heights and if the heights are the same, the areas are proportional to their base lengths.



- Heights of the $\triangle GBS$ & $\triangle OBS$ are the same $\Rightarrow \frac{[\triangle OBS]}{[\triangle GBS]} = \frac{OS}{GS} = \frac{3}{1} \Rightarrow [\triangle OBS] = 3 * [\triangle GBS] = 15$
- $\triangle OAS, \triangle OBS$ & $\triangle OCS$ are having the same base $\Rightarrow$ their areas are in the ratios of their heights $\Rightarrow$ sum of any smallest two areas = largest area. (from **2**)
- The possible values of $[\triangle OAS]$ are $[\triangle OBS] + [\triangle OCS] = 17$ and $|[\triangle OBS] - [\triangle OCS]| = 13$
- Hence the required answer is $17 + 13 = \boxed{30}$.

Problem 18 A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(a,b) = \left[\lim_{n\to\infty}\left(1 + \frac{b+a}{2!} + \frac{b^2+ab+a^2}{3!} + \frac{b^3+b^2a+ba^2+a^3}{4!} + \cdots \frac{b^{n-1}+b^{n-2}a+\cdots a^{n-1}}{n!}\right)\right]$$

Find

$$\text{fizz}\left(4125\cosh^{-1}\left(2024\left(f(2024,-2024) + \frac{f(-2024,-2024)}{2024}\right)\right)\right)$$

where fizz$(n) = n$ if n is a single digit number. Otherwise, fizz(n) = fizz(sum of the digits of n).

Solution We are given

$$f(a,b) = \left[\lim_{n\to\infty}\left(1 + \frac{b+a}{2!} + \frac{b^2+ab+a^2}{3!} + \frac{b^3+b^2a+ba^2+a^3}{4!} + \cdots \frac{b^{n-1}+b^{n-2}a+\cdots a^{n-1}}{n!}\right)\right]$$

Multiplying and dividing $(b-a)$ (assuming that $a \neq b$)

$$f(a,b) = \frac{1}{(b-a)}\left[\lim_{n\to\infty}\left((b-a) + \frac{b^2-a^2}{2!} + \frac{b^3-a^3}{3!} + \frac{b^4-a^4}{4!} + \cdots + \frac{b^n-a^n}{n!}\right)\right] = \frac{e^b-e^a}{b-a}$$

If $a = b$ the function simply reduces to

$$f(a,a) = \lim_{n\to\infty}\left(1 + \frac{a}{1!} + \frac{a^2}{2!} + \frac{a^3}{3!}\cdots\frac{a^n}{n!}\right) = e^a$$

Therefore $f(-2024,-2024) = e^{-2024}$ and $f(2024,-2024) = \dfrac{e^{2024}-e^{-2024}}{4048}$.

$$\implies \cosh^{-1}\left(2024\left(f(2024,-2024) + \frac{f(-2024,-2024)}{2024}\right)\right) = \cosh^{-1}\left(\frac{e^{2024}+e^{-2024}}{2}\right) = 2024$$

The function 'fizz' is simply the (repeated) sum of the digits of a number until we are left with one digit. This is equivalent to the 'modulo 9' operation. Therefore fizz$(2024 \times 4125) = 2024 \times 4125 \bmod 9 = (2025-1) \times (4122+3) \bmod 9 = -3 \bmod 9 = \boxed{6}$.

Problem 19 Let A, B be two matrices containing only real valued entries of order 7 satisfying the condition $|AB + (3 + 4i)I| = 0$. What is value of $|BA + (3 - 4i)I|$? Both A and B are invertible.

Solution We are given $|AB + (3 + 4i)I| = 0$. This implies that $-3 - 4i$ is an eigenvalue of the matrix AB (that is how we derive the characteristic equation in the first place). The eigenvalues of AB and BA are the same. This is shown below. Let $\mathbf{v}$ be an eigenvector of AB with an eigenvalue λ. Therefore

$$AB\mathbf{v} = \lambda\mathbf{v}$$

Left multiplying by B, we get

$$BAB\mathbf{v} = B\lambda\mathbf{v} = \lambda B\mathbf{v}$$

$$\implies BA(B\mathbf{v}) = \lambda(B\mathbf{v})$$

This means that $B\mathbf{v}$ is an eigenvector of BA with the same eigenvalue λ. We also note that the eigenvalues of AB must occur in conjugate pairs. This follows because A and B are both real valued, so the characteristic equation of AB has only real coefficients. Therefore $-3+4i$ is also an eigenvalue of AB. So $-3 + 4i$ is also an eigenvalue of BA. Therefore $|BA + (3 - 4i)I| = \boxed{0}$.

Problem 20 Let S be a set containing the positive values of $\alpha \leq 10\pi$ such that

$$\left| \int_{\Omega} \frac{1}{1 - z\cosh(iz)} dz \right|$$

is minimum where $\Omega = \{\, z : z = e^{i\theta} \ \forall \ \theta \in (0, \alpha] \,\}$. Find $\left[\sum_{\alpha \in S} \alpha \right]$. $[a]$ denotes the greatest integer less than or equal to a.

Solution We have $\cosh(iz) = \dfrac{e^{iz} + e^{-iz}}{2} = \cos(z)$

The given integral can be represented by the following sum

$$\left| \int_{\Omega} \frac{1}{1 - z\cos(z)} dz \right| = \left| \int_{\Omega} (1 + z\cos(z) + (z\cos(z))^2 + \dots)\, dz \right|$$

This is because $z\cos(z) \neq 1$ for $|z| = 1$. The integrand is analytic, therefore for $\alpha = 2n\pi$

$$\int_{\Omega} 1\, dz = 0 \qquad \implies \alpha = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$$

Therefore the answer is $\left[2 \times \dfrac{5 \times 6}{2}\pi \right] = \boxed{94}$.

Problem 21 Consider a prime number p ($p \neq 2$) such that a triangle has two sides of length p and $\dfrac{p^2 - 1}{2}$. Let the largest side of the triangle be x such that

$$x^{p-1} \equiv 1 \pmod{p}$$

Determine the angle (in degrees) made by the given smaller two sides of the triangle.

Solution **Bonus**: printing error. This question has a nice solution if the condition were

$$x^{p-1} \equiv 2x \pmod{p}$$

Due to Fermat's Little Theorem,

$$2x \equiv 1 \pmod{p}$$

The maximum possible value of the third side x cannot exceed $p + \frac{p^2-1}{2}$ (Triangle Inequality). Clearly x has to be of the form $\frac{np+1}{2}$ where n is at least p (x is the largest side). If $n = p+1$,

$$x = \frac{p^2+1}{2} + \frac{p}{2}$$

which is not an integer. Now let us try $n = p+2$. We can see that x violates the triangle inequality. Hence *nhas to be p*. This means that

$$x = \frac{p^2+1}{2}$$

This implies that the sides of the triangle satisfy Pythogoras' Theorem (Baudhāyana Śulbasūtra, c. 1000 BCE). Hence the angle made by the smaller two sides of the triangle in degrees is $\boxed{90}$.

Problem 22 Consider a square matrix A of order 3, with three unique eigenvalues $\lambda_1,\ \lambda_2\ and\ \lambda_3$. The corresponding eigenvectors are $\vec{x}_1,\ \vec{x}_2\ and\ \vec{x}_3$ make a matrix P . Let α be the determinant of the matrix S such that

$$S = PAP^{-1}$$

Consider the following

$$\epsilon = \lambda_3^2 + \alpha$$

$$\delta = \lambda_1^2 + \alpha$$

ϵ when multiplied by the eigenvalue of A^{-1} corresponding to $\vec{x}_3$ gives the value 10,038 and δ when multiplied by the eigenvalue of A^{-1} corresponding to $\vec{x}_1$ gives the value 9989. Find the number of possible integral solutions of $(\lambda_1, \lambda_2, \lambda_3)$.

Solution **Bonus**: self contradiction. A is not invertible because the only solution to the system of equations involves a 0 eigenvalue. This implies that the determinant is 0.

Problem 23 A positive integer k is said to be interesting if there exists a partition of {1,2,3,4,5,6,7,8, ... ,30} into disjoint proper subsets (not necessarily only 2) such that the sum of numbers in each subset of the partition is k. How many interesting numbers exist?
Definition: A partition of a set is a grouping of its elements into non-empty subsets in such a way that every element is included in *exactly one subset.*

Solution The sum of the all the elements in the set is

$$1 + 2 + 3 + 4 + 5 + ... + 29 + 30 = \frac{30 \times 31}{2} = 465$$

Every interesting number k will be a factor of the 465. This is because each partition must have the same sum k and this multiplied by the number of partitions must yield 465. The factors of 465 are 1, 3, 5, 15, 31, 93, 165 and 465. Also, k must be more than or equal to 30 since 30 should exist in one partition. k cannot be 465 because that would not form any partition. Hence k can take the values 31, 93 and 165. This would form 15, 5 and 3 subsets respectively, all of which are possible. Hence, the number of good numbers are $\boxed{3}$.

Problem 24 Let N=6+66+666+6666+........ +66666666.....66 where there are 2025 6's in the last term of the sum. How many times does the digit 4 occur in N when written in base 10?

Solution Consider the number N,

$$N = 6 + 66 + 666 + 6666 + \ldots + \overbrace{6666...666}^{2025 \text{ times}}$$

The above number can be rewritten as

$$N = \frac{6}{9}(9 + 99 + 999 + 9999 + \ldots + \overbrace{9999...999}^{2025 \text{ times}})$$

$$= \frac{6}{9}((10-1) + (100-1) + (1000-1) + \ldots.. + (10^{2025} - 1))$$

$$= \frac{6}{9}(10 + 100 + 1000 + \ldots.. + 10^{2025} - 2025)$$

Using the formula for the sum of a Geometric Progression, the expression simplifies to

$$\implies N = \frac{60}{81} \times \overbrace{999999999......99}^{2025 \text{ times}} - \frac{6 \times 2025}{9}$$

Dividing by 9 generates a pattern which repeats every three digits. Divide 1,111,111 by 3 (and then multiply by 2) to see the pattern of '740's.

$$\implies N = \overbrace{740740740....740740}^{675 \text{ times}} - 1350 = 740740740..73930$$

Therefore '4' occurs 673 times in the given sequence. The answer is $\boxed{673}$.

Problem 25 One day, Karthikeya and Prad decided to play an integration game. They considered a function $g : [0, 1] \to \mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers) which satisfies the conditions of the problem stated below.

$$\text{Karthikeya's Score} = K = \int_0^1 x^2 g(x) dx$$

$$\text{Prad's Score} = P = \int_0^1 x(g(x))^2 dx$$

You have been told that Karthikeya has won the game ($K > P$). Your task as a judge is to now figure out the maximum possible difference between Karthikeya's and Prad's scores ($K - P$). If this difference is denoted by λ_m then answer $2032\lambda_m$.

Solution Since we know that Karthikeya has won the game, we shall try to maximize $K - P$. This is

$$\int_0^1 (x^2.g(x))dx - \int_0^1 x.(g(x))^2 dx$$

Intuitively we want to get rid of the dependence of $g(x)$ because we do not know its properties. Completing the square for $g(x)$ we get

$$\int_0^1 -(\sqrt{x}g(x))^2 - 2\sqrt{x}g(x)\frac{x^{\frac{3}{2}}}{2} - \left(\frac{x^{\frac{3}{2}}}{2}\right)^2 + \left(\frac{x^{\frac{3}{2}}}{2}\right)^2 dx$$

$$= \int_0^1 \left(\frac{x^{\frac{3}{2}}}{2}\right)^2 - \left(\sqrt{x}g(x) + \frac{x^{\frac{3}{2}}}{2}\right)^2 dx$$

The maximum value of this expression is obtained when the term inside the square is 0 - when $g(x) = \frac{x}{2}$. Hence the maximum value is

$$\lambda_m = \int_0^1 \frac{x^3}{4} dx = \frac{1}{16}$$

Therefore the value of $2032\lambda_m = \boxed{127}$.

Problem 26 Atreya is initially (when $t = 0$) at the point $(5, 10)$. At a given time $t \in \mathbb{N}$ seconds, Atreya's position is denoted by (x_t, y_t). He moves one step every second in the plane according to the following rules.

- The x-component of his step is y_t.
- The y-component of the step is $x_t - y_t$.

If $\lim_{t\to\infty} \left(\frac{y_t}{x_t}\right) = \frac{\sqrt{a} - b}{c}$, where a, b and c are co-prime, Find $a + b + c$.

Solution The step size at time t is given by $x_{t+1} - x_t$. Therefore the two conditions translate to the below equations.

$$x_{t+1} - x_t = y_t$$

$$y_{t+1} - y_t = x_t - y_t$$

Simplifying, this reduces to

$$x_{t+1} = x_t + y_t$$

$$y_{t+1} = x_t$$

Method 1
On observation we see a clever pattern. x_t, y_t are adjacent numbers of a Fibonacci sequence (Piṅgala, c. 300 BCE) with a different initial point. Irrespective of the initial conditions the ratio of $(n + 1)^{th}$ to n^{th} numbers converges to ϕ (golden ratio) as $n \to \infty$. Therefore

$$\lim_{t\to\infty} \left(\frac{y_t}{x_t}\right) = \frac{1}{\phi} = \frac{\sqrt{5} - 1}{2}$$

Comparing with the form given in the question the answer is $a + b + c = 5 + 1 + 2 = \boxed{8}$.

Method 2
Write the system of linear equations as below

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

The state $t+2$ depends on $t+1$ similar to the above system of equations which can then be represented in terms of the state at t.

$$\begin{pmatrix} x_{t+2} \\ y_{t+2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Therefore the state at time n depends on the initial state as below

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Perron-Frobenius Theorem: A matrix (conditions apply) will transform all points/vectors along the eigenvector with largest eigenvalue.

Therefore we find the largest eigenvalue and then find the corresponding eigenvector and then take the ratio of its components to get the required answer, as given below:

The largest eigenvalue is $\lambda_0 = \dfrac{\sqrt{5}+1}{2}$. Therefore the ratio of components of the eigenvector is

$$\frac{1}{\lambda_0} = \frac{\sqrt{5}-1}{2}$$

Method 3

Divide the first equation by the second. As t approaches infinity this ratio converges to L.

$$\begin{aligned} \frac{y_{t+1}}{x_{t+1}} &= \frac{x_t}{x_t + y_t} \\ &= \frac{1}{1 + \frac{y_t}{x_t}} \\ \implies L &= \frac{1}{1+L} \\ \implies L &= \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2} \end{aligned}$$

L is positive so only the first solution makes sense. Therefore the answer is $\boxed{8}$.

Problem 27 In n-dimensional space we consider two figures.

- A hyper-sphere given by the following equation:

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = \left(\frac{n+1}{2}\right)^2$$

- A hyper-ellipsoid given by the following equation:

$$\frac{x_1^2}{1^2} + \frac{x_2^2}{2^2} + \frac{x_3^2}{3^2} + \cdots + \frac{x_n^2}{n^2} = 1$$

Let $L = \lim_{n\to\infty} \left(\dfrac{S}{E}\right)^{\frac{2}{n}}$, where S and E are the hyper-volumes of the hyper-sphere and hyper-ellipsoid respectively. Find the value of $[4L]$. $[a]$ denotes the greatest integer less than or equal to a.

Solution The hyper-volume of the hyper-sphere is proportional to $\left(\frac{n+1}{2}\right)^n$ and that of hyper ellipsoid is proportional to $n!$. Why? The volume of a sphere is proportional to r^3 and the volume of an ellipsoid is proportional to abc where r is the radius of the sphere and a,b and c are the dimensions of the ellipsoid. The proportionality constants are also the same because the ellipsoid can always be scaled to a sphere (When $a = b = c = r$, then the ellipsoid is a sphere). Therefore

$$L = \lim_{n\to\infty}\left(\frac{S}{E}\right)^{\frac{2}{n}} = \lim_{n\to\infty}\left(\frac{\left(\frac{n+1}{2}\right)^n}{n!}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n+1}{2}\right)\frac{1}{n!^{\frac{1}{n}}}$$

Using Stirling's approximation for very large n:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Substituting we get

$$\lim_{n\to\infty}\left(\frac{n+1}{2}\right)\frac{1}{\frac{n}{e}\left(\sqrt{2\pi n}\right)^{\frac{1}{n}}}$$

Using the fact that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ and $\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \;\forall\; a > 0$, $\left(\lim_{n\to\infty}(2\pi)^{\frac{1}{n}} = 1\right)$:

$$\lim_{n\to\infty}\left(\frac{\left(\frac{n+1}{2}\right)^n}{n!}\right)^{\frac{1}{n}} = \frac{e}{2}$$

$$L = \lim_{n\to\infty}\left(\frac{\left(\frac{n+1}{2}\right)^n}{n!}\right)^{\frac{2}{n}} = \left(\frac{e}{2}\right)^2$$

$$\therefore [4L] = \left[e^2\right] = \boxed{7}$$

Problem 28 Define I_1 and I_2 as follows:

$$I_1 = \int_0^\infty \frac{e^{-x^2}}{2024 + x^2}\,dx$$

$$I_2 = \int_1^\infty e^{-2024x^2}\,dx$$

Find $\log_e\left(\frac{I_1\sqrt{\pi}}{I_2}\right)$. The below integral might be useful:

$$\int_0^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{4a}} \quad \text{for } a > 0.$$

Solution **Bonus**: answer is not an integer.

$$I_1 = \int_0^\infty \frac{e^{-x^2}}{2024 + x^2}\,dx$$

Define $I(a)$:

$$I(a) = \int_0^\infty \frac{e^{-a(2024+x^2)}}{2024+x^2}\,dx$$

$$I'(a) = \int_0^\infty \frac{e^{-a(2024+x^2)} \cdot (-2024-x^2)}{2024+x^2}\,dx$$
$$= -e^{-2024a} \int_0^\infty e^{-ax^2}\,dx$$

$$I'(a) = -e^{-2024a}\sqrt{\frac{\pi}{4a}}$$

$$I_1 = e^{2024} \int_0^\infty \frac{e^{-(2024+x^2)}}{2024+x^2}\,dx$$
$$= e^{2024} I(1)$$
$$= e^{2024}\left[I(1) - \lim_{b\to\infty} I(b)\right]$$
$$= -e^{2024} \int_1^\infty -\sqrt{\frac{\pi}{4a}} e^{-2024a}\,da$$
$$= e^{2024} \int_1^\infty \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-2024a}\,da$$
$$I_1 = e^{2024} \int_1^\infty \sqrt{\pi} e^{-2023t^2}\,dt$$

where $a = t^2, da = 2\sqrt{a}dt$. Therefore

$$I_1 = e^{2024}\sqrt{\pi} I_2$$

Therefore $\log_e\left(\frac{I_1\sqrt{\pi}}{I_2}\right) = \log_e(e^{2024}\pi) = \boxed{2024 + \log_e(\pi)}$.

Problem 29 Consider a point $P(39\sqrt{2}, 89\sqrt{2})$ and a conic section $C \equiv x^2 - 4\sqrt{2}x + y^2 + 4\sqrt{2}y + 2xy = 0$. Let T be the number of tangents that can be drawn from point P to C , and let N be the number of normals to C that pass through P, then calculate $T^N + N^T$.

Solution The given conic C can be written as $Y^2 = 4aX$ where:

$$Y = \frac{x+y}{\sqrt{2}}$$
$$X = \frac{x-y}{\sqrt{2}}$$
$$a = 1$$

Therefore, $P(x, y) = P(39\sqrt{2}, 89\sqrt{2})$ in the transformed coordinates is $P'(X, Y) = P'(-50, 128)$. The number of tangents from P' to $C' \equiv Y^2 = 4aX$ is $T = 2$ since P' is outside the parabola C'. The normals to the parabola $Y^2 = 4aX$ have the form $Y = mX - 2am - am^3$. On substituting the values of X, Y, a, the following cubic equation is obtained:

$$m^3 + 52m + 128 = 0$$

Let the LHS of the above cubic equation in m be $f(m)$. It can be seen that $f'(m) = 3m^2 + 52 > 0$ for all m. This implies that $f(m)$ is a monotonically increasing function and therefore has only 1 zero. Therefore only $N = 1$ normal can pass through this point.

$$\therefore T^N + N^T = 2^1 + 1^2 = \boxed{3}$$

Problem 30 Define $I = \int\limits_{-\infty}^{0} e^{2025x}(1 - e^x)^{2024}\, dx$. Calculate $\left[\binom{4046}{2023} \times I\right]$ where $[x]$ is the greatest integer less than or equal to x.

Solution Consider the integral given. Let $B(m, n)$ denote the Beta function. Let $\Gamma(n)$ denote the Gamma function.

$$\begin{aligned}
I &= \int_0^1 e^{2025x}(1 - e^x)^{2024}\, dx \\
&= \int_0^1 t^{2025}(1 - t)^{2024} \frac{dt}{t} \quad \text{where } e^x = t \\
&= \int_0^1 t^{2024}(1 - t)^{2024}\, dt \\
&= B(2025, 2025) \\
&= \frac{\Gamma(2025)\Gamma(2025)}{\Gamma(2025 + 2025)} \\
&= \frac{2024! \cdot 2024!}{4049!} \\
\implies \left[I \times \binom{4046}{2023}\right] &= \left[\frac{2024^2 \times 4046!}{4049!}\right] \\
&= \left[\frac{2024 \times 2024}{4047 \times 4048 \times 4049}\right]
\end{aligned}$$

This implies that

$$\left[I \times \binom{4046}{2023}\right] = \boxed{0}$$