Srinivasa Ramanujan Mathematics Competition 2024

Round 1 (Prelims Question Paper)

Conducted on: 21st July 2024

1. Pradyumnan and Atreya play a game on a 7×7 matrix (currently empty), placing *unique* numbers from 1 to 49 (both inclusive) in each location. Pradyumnan starts, placing a number between 1 and 49 in any cell. Atreya follows, placing another number in a different cell. They alternate turns until the matrix is filled with all 49 numbers. Pradyumnan wins if the determinant of the resulting matrix is divisible by 49, and Atreya wins otherwise. Both play optimally and want to win. Who wins? Enter 1 if Atreya wins and 2 if Pradyumnan wins.

2. The polynomials P(x), Q(x), R(x), S(x) satisfy the equation,

$$P(x^5) + \frac{x}{3}Q(x^{10}) + \frac{x^2}{3^2}R(x^{15}) = \left(1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \frac{x^4}{3^4}\right) S(x^3) + (3x+2)^2$$

   Find the value of

$$P(3^5) + Q(3^{10}) + R(3^{15})$$

3. Let AB be a chord of a circle S. P is any point on the circle. It is given that the perpendicular distance between P and AB is 6 units. If the perpendicular distance between P and the tangent at A is 4 units, what is the perpendicular distance between P and the tangent at B?

4. You are given two hourglasses which can measure 11 minutes and 7 minutes (the sand in each of the hourglasses takes that much time to run out). You don't have access to any other clocks and you cannot determine the amount of sand in each hourglass at an arbitrary time. You can claim that you can measure a particular time interval t minutes if and only if you can specify a 'start' and an 'end' time and the time interval in between these times equals t minutes. How many of the below time intervals can be measured using the above definition?

   - 17 minutes
   - 9 minutes
   - 1 minute

5. Consider an infinite row of cells (empty boxes). Each cell can be in either of two states: *on* or *off*. These states can change over time (discrete steps). Consider a process called *Infection* defined below:

   - If neither of the neighbour cells of a particular cell are infected, that cell recovers in the next time step (turns *off*).
   - If exactly one of the neighbour cells are infected, the cell under consideration also becomes infected in the next time step (turns *on*).
   - If both the neighbour cells are infected, the cell under consideration **recovers** in the next time step (turns *off*).

   Initially one cell is infected (call this the *origin* cell). Call this time t = 0. Denote the cell to the immediate right of the *origin* cell to be the *friend* cell. Consider the states of the *friend* cell at time steps t = 1, 63, 185, 273, 510. How many times among the 5 time steps given is the *friend* cell infected or equivalently, *on*?

6. One day Nikhil decided that the Greatest Integer Function (hereafter referred to as GIF) is too boring. The GIF of a *real* number x is the greatest integer less than or equal to x.

   - Nikhil extends this definition to the complex numbers as $[a + bi] = [a] + [b]i$.
   - Karthikeya, who loves writing complex numbers in their polar form is shocked. He defines $[r \exp(i\theta)] = [r] \exp(i[\theta])$ where θ is represented in *degrees*.

   A complex number is called *nice* if its Nikhil GIF and Karthikeya GIF are the same. A *nice GIF* is defined as the GIF of a nice number. For complex numbers with real and imaginary parts a, b such that $a \in [0, 13)$ and $b \in [0, 13)$ how many possible *nice GIFs* exist?

7. Consider the following integral-

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\cos^2(x-y) - 2\cos(x-y)\cos(x+y) + \cos^2(x+y)}{x^2y^2} dxdy$$

   Find $[\sqrt{I}]$. $[a]$ is defined as the greatest integer less than or equal to a.

8. Evaluate the absolute value of the sum of the components of the resultant matrix/vector

$$\int \begin{bmatrix} 3y & 3x \\ 4 & 2y \end{bmatrix} d \begin{bmatrix} x \\ y \end{bmatrix}$$

   from (3, 4) to (0, 5) along the circle centered at the origin anticlockwise.

9. Consider two sets A and B defined below.

   - Set A contains *all* vectors of the form $(a_1, a_2, a_3, ...)$ where $a_i \in \mathbb{N} + \{0\}$ and $\exists n \in \mathbb{N} : a_k = 0 \ \forall \ k > n$. In other words the number of choices of each element is infinite (all non-negative integers) but the number of such elements is finite.
   - Set B contains *all* vectors of the form $(b_1, b_2, b_3, ...)$ where $b_i \in \{0, 1\}$. Now the number of choices for each element is finite (2) but the number of such elements is infinite.

   Let $|X|$ denote the cardinality of a set X. If $|A| > |B|$, answer 0. If $|A| = |B|$ answer 1. Otherwise answer 2.

10. We define $T(A, B)$ to be the number of ways in which A candies can be distributed among B children when every child gets at least one candy. Find $\sum_{i=3}^{12} T(i + 1, 3)$.

11. Consider two sets. The first set (denoted by C_1) contains all points (x, y) which satisfy the equation $x^2 + \frac{y^2}{4} = k_1$. We will refer to this equation as the C_1 curve henceforth. The second set (denoted by C_2) contains all points (x, y) satisfying an equation referred to as the C_2 curve. At every intersection point of the C_1 curve and the C_2 curve, the tangents to the curves at these points have x and y intercepts at x_1, y_1, x_2, y_2 respectively. These intercepts satisfy the condition $y_1y_2 = 4x_1x_2$. If $(11, 21)$ and $(3, \alpha)$ both belong to C_2, what is the value of α?

12. What is the expected number of throws of a die until we get the total (cumulative) sum to have remainder 3 when divided by 7?

13. If $H = \begin{bmatrix} 3 & 7 & 1 \\ -5 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{x} = [x_1, x_2, x_3]^T$ find maximum value of $\mathbf{x}^T H \mathbf{x}$ given that $||\mathbf{x}||$=1.

14. You are given that f(x)=$\begin{cases} \sqrt{16-(x+8)^2} & -8 \leq x \leq -4 \\ x^2-16 & -4 \leq x \leq 4 \\ \sqrt{16-(x-8)^2} & 4 \leq x \leq 8 \end{cases}$. If the minimum value of the integral

$$\int_{-8}^{8} |f(x)-k|dx$$

over all values of k is x, find $[x]$. $[a]$ denotes the greatest integer less than or equal to a.

15. You are given an ellipse $E := \frac{x^2}{4} + \frac{y^2}{9} = 1$ with center O. The tangents of E at Q and R intersect at a point P such that $\frac{[\triangle PQR]}{[\triangle OQR]} = 8$. The locus of P is given by S. Define $F(A)$ to be the perimeter of A. Find the value of $\frac{F(S)}{F(E)}$. Note that $[\triangle ABC]$ denotes the area of the $\triangle ABC$.

16. In $\triangle ABC, AB = 7, AC = 2$ and $II_A = 6$ where I is the incentre of the triangle and I_A is the excentre opposite to the vertex A. Find the value of $92(BC)^2$.

17. In $\triangle ABC$, $[\triangle GBS] = 5, [\triangle OCS] = 2$ then find the sum of all possible values of $[\triangle OAS]$, where G is the centroid, S is the circumcenter and O is the orthocenter. Note that $[\triangle ABC]$ denotes the area of the $\triangle ABC$.

18. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(a,b) = \left[\lim_{n\to\infty}\left(1 + \frac{b+a}{2!} + \frac{b^2+ab+a^2}{3!} + \frac{b^3+b^2a+ba^2+a^3}{4!} + \cdots \frac{b^{n-1}+b^{n-2}a+\cdots a^{n-1}}{n!}\right)\right]$$

Find

$$\text{fizz}\left(4125\cosh^{-1}\left(2024\left(f(2024,-2024)+\frac{f(-2024,-2024)}{2024}\right)\right)\right)$$

where fizz$(n) = n$ if n is a single digit number. Otherwise, fizz(n) = fizz(sum of the digits of n).

19. Let A, B be two matrices containing only real valued entries of order 7 satisfying the condition $|AB + (3+4i)I| = 0$. What is value of $|BA + (3-4i)I|$? Both A and B are invertible.

20. Let S be a set containing the positive values of $\alpha \leq 10\pi$ such that

$$\left|\int_{\Omega} \frac{1}{1 - z\cosh(iz)} dz\right|$$

is minimum where $\Omega = \{\ z : z = e^{i\theta}\ \forall\ \theta \in (0,\alpha]\ \}$. Find $\left[\sum_{\alpha\in S} \alpha\right]$. $[a]$ denotes the greatest integer less than or equal to a.

21. Consider a prime number p $(p \neq 2)$ such that a triangle has two sides of length p and $\frac{p^2-1}{2}$. Let the largest side of the triangle be x such that

$$x^{p-1} \equiv 1 \pmod{p}$$

Determine the angle (in degrees) made by the given smaller two sides of the triangle.

22. Consider a square matrix A of order 3, with three unique eigenvalues λ_1, $\lambda_2 and \lambda_3$. The corresponding eigenvectors are $\vec{x}_1$, $\vec{x}_2\, and\, \vec{x}_3$ make a matrix P . Let α be the determinant of the matrix S such that

$$S = PAP^{-1}$$

Consider the following

$$\epsilon = \lambda_3^2 + \alpha$$
$$\delta = \lambda_1^2 + \alpha$$

ϵ when multiplied by the eigenvalue of A^{-1} corresponding to $\vec{x}_3$ gives the value 10,038 and δ when multiplied by the eigenvalue of A^{-1} corresponding to $\vec{x}_1$ gives the value 9989. Find the number of possible integral solutions of $(\lambda_1, \lambda_2, \lambda_3)$.

23. A positive integer k is said to be interesting if there exists a partition of {1,2,3,4,5,6,7,8, ... ,30} into disjoint proper subsets (not necessarily only 2) such that the sum of numbers in each subset of the partition is k. How many interesting numbers exist?
Definition: A partition of a set is a grouping of its elements into non-empty subsets in such a way that every element is included in *exactly one subset.*

24. Let N=6+66+666+6666+. . . . . +66666666. . . ..66 where there are 2025 6's in the last term of the sum. How many times does the digit 4 occur in N when written in base 10?

25. One day, Karthikeya and Prad decided to play an integration game. They considered a function $g : [0, 1] \to \mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers) which satisfies the conditions of the problem stated below.

$$\text{Karthikeya's Score} = K = \int_0^1 x^2 g(x)dx$$

$$\text{Prad's Score} = P = \int_0^1 x(g(x))^2 dx$$

You have been told that Karthikeya has won the game $(K > P)$. Your task as a judge is to now figure out the maximum possible difference between Karthikeya's and Prad's scores $(K - P)$. If this difference is denoted by λ_m then answer $2032\lambda_m$.

26. Atreya is initially (when $t = 0$) at the point $(5, 10)$. At a given time $t \in \mathbb{N}$ seconds, Atreya's position is denoted by (x_t, y_t). He moves one step every second in the plane according to the following rules.

- The x-component of his step is y_t.
- The y-component of the step is $x_t - y_t$.

If $\lim_{t\to\infty} \left(\frac{y_t}{x_t}\right) = \frac{\sqrt{a} - b}{c}$, where a, b and c are co-prime, Find $a + b + c$.

27. In n-dimensional space we consider two figures.

- A hyper-sphere given by the following equation:

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = \left(\frac{n+1}{2}\right)^2$$

- A hyper-ellipsoid given by the following equation:

$$\frac{x_1^2}{1^2} + \frac{x_2^2}{2^2} + \frac{x_3^2}{3^2} + \cdots + \frac{x_n^2}{n^2} = 1$$

Let $L = \lim_{n\to\infty} \left(\frac{S}{E}\right)^{\frac{2}{n}}$, where S and E are the hyper-volumes of the hyper-sphere and hyper-ellipsoid respectively. Find the value of $[4L]$. $[a]$ denotes the greatest integer less than or equal to a.

28. Define I_1 and I_2 as follows:

$$I_1 = \int_0^\infty \frac{e^{-x^2}}{2024 + x^2}\,dx$$

$$I_2 = \int_1^\infty e^{-2024x^2}\,dx$$

Find $\log_e\left(\frac{I_1\sqrt{\pi}}{I_2}\right)$. The below integral might be useful:

$$\int_0^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{4a}} \quad \text{for } a > 0.$$

29. Consider a point $P(39\sqrt{2}, 89\sqrt{2})$ and a conic section $C \equiv x^2 - 4\sqrt{2}x + y^2 + 4\sqrt{2}y + 2xy = 0$. Let T be the number of tangents that can be drawn from point P to C , and let N be the number of normals to C that pass through P, then calculate $T^N + N^T$.

30. Define $I = \int_{-\infty}^{0} e^{2025x}(1 - e^x)^{2024}\,dx$. Calculate $\left[\binom{4046}{2023} \times I\right]$ where $[x]$ is the greatest integer less than or equal to x.