Advent Calendar 2024

Solutions

Mathematics Club

IIT MADRAS

v1

§ Pentaweek 1 (01-05 Dec)

Day 1 | Let S be a non-empty finite set of positive integers satisfying:

$$\left(\frac{1}{i} + \frac{1}{j}\right) \times \text{LCM}\,(i, j) \in S \quad \forall\, i, j \in S$$

Find the sum of the cardinalities of all possible S.

Solution:

Notice that the given expression is the same as $\dfrac{i+j}{\text{GCD}(i,j)}$ because $\text{LCM}\,(i,j) \times \text{GCD}\,(i,j) = ij$.
If we set $i = j$ we observe that the expression simply reduces to 2. Thus, if S were non-empty then it must contain 2.

Now pick an element k $(\neq 2)$ from S. Since S contains 2, S must also contain $\dfrac{2+k}{\text{GCD}\,(2,k)}$.

If k were odd, then $\frac{2+k}{\text{GCD}(2,k)} = 2 + k \in S$. This implies that if S contains an odd number, then it must contain every following odd number as well, thereby making S infinite. Thus, a finite S cannot contain any odd numbers. If k were even, then $\frac{2+k}{\text{GCD}(2,k)} = 1 + \frac{k}{2} \in S$. With $k' = 1 + \frac{k}{2}$, we need $\frac{2+k'}{\text{GCD}(2,k')} \in S$. Continuing this procedure, we are bound to eventually hit an odd number which would make S infinite.

Thus, the only possible $S = \{2\}$ and the sum of cardinalities of all possible S is 1.

Day 2 | Pranjal has a challenge for Achintya. Let ε denote the non-terminating decimal expansion of a fraction (Take $\frac{1}{4}$ as an example, its ε is $0.24\overline{9}$). Pranjal asks him to write the ε of $\frac{1}{k}$, for any natural number $k > 1$. Achintya is then required to find the number of digits in the non-repeating part of ε. (It is 2 $(0.\mathbf{24}\overline{9})$ in the case of $\frac{1}{4}$). Can you help him?
What would be the number of digits in the non-repeating part of ε for k $= 2^{1234} \times 3^{2345} \times 4^{3456} \times 5^{4567}$

Solution:

We can write $\frac{1}{n} = 0.a_1a_2\ldots a_r\overline{b_1b_2\ldots b_s}$, where a and b represent the numbers formed by the non-repeating and repeating parts, respectively:

$$a = a_1a_2\ldots a_r, \quad b = b_1b_2\ldots b_s,$$

and r is the length of the non-repeating part.

Thus,

$$\frac{1}{n} = \frac{1}{10^r}\left(a + \sum_{k\geq 1} \frac{b}{(10^s)^k}\right).$$

The infinite sum can be simplified using the formula for a geometric series:

$$\frac{1}{n} = \frac{1}{10^r}\left(a + \frac{b}{10^s - 1}\right).$$

Rearranging, we get:

$$10^r(10^s-1)=n\big((10^s-1)a+b\big).$$

Now, for any prime p, let $v_p(n)$ denote the maximum power of p that divides n. By definition, $p^{v_p(n)}$ divides n, but $p^{v_p(n)+1}$ does not.

From the equation above, we deduce that

$$r\geq \max(v_2(n),v_5(n)),$$

Suppose $r>\max(v_2(n),v_5(n))$. Then 10 divides $b-a$, which implies that the last digits of a and b are equal, i.e., $a_r=b_s$. In this case,

$$\frac{1}{n}=0.a_1a_2\ldots a_{r-1}\overline{b_sb_1b_2\ldots b_{s-1}}.$$

This contradicts the definition of r, as r is supposed to represent the length of the non-repeating part. Therefore,

$$r=\max(v_2(n),v_5(n)).$$

Day 3 | Let $f:\mathbb{N}\to\mathbb{R}$ which satisfies the following property:
$f(1)+2^2f(2)+3^3f(3)+4^4f(4).....n^nf(n)=\left(\frac{n^{n+1}}{3}+\frac{n^n}{2}+\frac{n^{n-1}}{6}\right)f(n)$.

Find $\log_{10}\left(\frac{f(100)}{f(1)}\right)$.

Solution:

$$f(1)+2^2f(2)+3^3f(3)+4^4f(4).....n^nf(n)=\left(\frac{n^{n+1}}{3}+\frac{n^n}{2}+\frac{n^{n-1}}{6}\right)f(n)$$

$$\implies \left(\frac{(n-1)^n}{3}+\frac{(n-1)^{n-1}}{2}+\frac{(n-1)^{n-2}}{6}\right)f(n-1)+n^nf(n)=\left(\frac{n^{n+1}}{3}+\frac{n^n}{2}+\frac{n^{n-1}}{6}\right)f(n)$$

$$\implies \left(\frac{(n-1)^n}{3}+\frac{(n-1)^{n-1}}{2}+\frac{(n-1)^{n-2}}{6}\right)f(n-1)=\left(\frac{n^{n+1}}{3}-\frac{n^n}{2}+\frac{n^{n-1}}{6}\right)f(n)$$

$$\implies \left(2(n-1)^n+3(n-1)^{n-1}+(n-1)^{n-2}\right)f(n-1)=\left(2n^{n+1}-3n^n+n^{n-1}\right)f(n)$$

$$\implies (n-1)^{n-2}\left(2(n-1)^2+3(n-1)+1\right)f(n-1)=n^{n-1}\left(2n^2-3n+1\right)$$

$$\implies (n-1)^{n-2}\left(2(n-1)^2+3(n-1)+1\right)f(n-1)=n^{n-1}\left(2n^2-3n+1\right)f(n)$$

$$\implies n(n-1)^{n-2}\left(2n-1\right)f(n-1)=n^{n-1}\left(2n-1\right)\left(n-1\right)f(n)$$

$$\implies \frac{f(n)}{f(n-1)}=\frac{(n-1)^{n-3}}{n^{n-2}}S\ \forall\ n\geq 3$$

By using standard telescoping (which may be rigorously proved using induction), we get:

$$\frac{f(n)}{f(3)}=\frac{3}{n^{n-2}}\ \forall\ n\geq 3$$

By brute forcing the values of $f(2)$ and further $f(3)$ in terms of $f(1)$ from the original equation, we get:

$$f(3) = \frac{f(1)}{3}$$

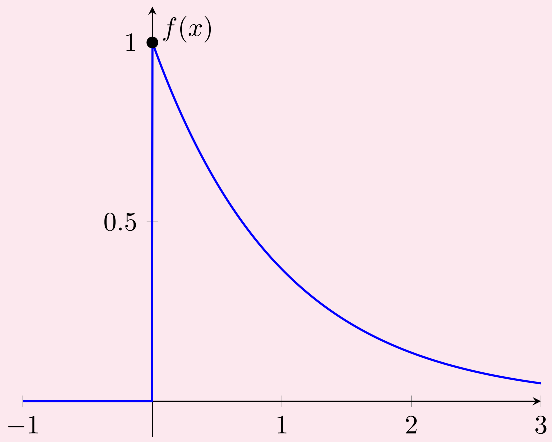
Thus,

$$\frac{f(n)}{f(1)} = \frac{1}{n^{n-2}}$$

and $\log_{10}\left(\frac{f(100)}{f(1)}\right) = \log_{10}\left(\frac{1}{100^{98}}\right) = -196$

Day 4 | Pranjal is at the point $(-1, 0)$. He can take a step of magnitude r only towards the right with probability density function (PDF) $f_R(r) = Ce^{-r}$ (where C is an appropriate constant to ensure that the generated PDF is valid).
At what point is he expected to be at after 1729 steps? Enter the x-value. (Hint: What should C be?)

Solution:



$$f(x) = \begin{cases} e^{-x} & \forall\ x \geq 0 \\ 0 & \forall\ x < 0 \end{cases}$$

$$\mathrm{P}(x'|x) = \begin{cases} e^{-(x'-x)} & \forall\ x' - x \geq 0 \\ 0 & \forall\ x' - x < 0 \end{cases}$$

$$\mathrm{P}(x'|x) = \begin{cases} e^{-(x'-x)} & \forall\ x' \geq x \\ 0 & \forall\ x' < x \end{cases}$$

After one step we get the p.d.f. $= \mathrm{P}_1(x) \begin{cases} e^{-x} & \forall\ x \geq 0 \\ 0 & \forall\ x < 0 \end{cases}$

After two steps we get the p.d.f. $= \mathrm{P}_2(x) = \int_{-\infty}^{+\infty} P_1\left(x|x'\right) f\left(x'\right) dx' = \begin{cases} xe^{-x} & \forall\ x \geq 0 \\ 0 & \forall\ x < 0 \end{cases}$

After n steps we get the p.d.f. $= \mathrm{P}_n(x) = \int_{-\infty}^{+\infty} P_{n-1}\left(x|x'\right) f\left(x'\right) dx' = \begin{cases} \frac{x^{(n-1)}}{(n-1)!}e^{-x} & \forall\ x \geq 0 \\ 0 & \forall\ x < 0 \end{cases}$

$E_n[X] = \int_{-\infty}^{+\infty} xP_n(x)\, dx = n$

Pranjal is expected to end up at 1729 from his initial position. But he starts from (-1,0).

Thus, Pranjal ends up at **1728**.

Day 5 | Find number of all combinations (a, b, c, p) of positive integers a,b,c and prime number p such that

$$2^a p^b = (p + 2)^c + 1$$

Assume $b \leq 5$ and $b \leq c$. You might want to have a look at P-adic valuation and Lifting the exponent.

Solution:

p must be odd as for p = 2, LHS $\not\equiv$ RHS mod 2 *Case 1*: Let us first check for cases with $b < c$

- *Case 1.1*: If c is odd:
  - p + 3 is even (i.e. $2|(p + 2) + 1$) so $V_2(RHS) = V_2(p + 3) = V_2(LHS) = a$ (Lifting the exponent is necessary for this step)
  - We can conclude that $p + 3 = (2^a) * (oddfactor)$ Thus $2^a \leq p + 3$
  - Now LHS $\leq p^{b+1} + 3 * p^b$ and RHS $= (p+2)^c + 1$. We will now try to show that $RHS > LHS$.
  - The smallest value that we can take for c is b + 1. Even in this case $RHS > LHS$ ($b > 0$ so $2(b + 1) > 3$). Thus there are no solutions
- *Case 1.2*: If c is even:
  - $V_2(RHS) = 1$ and $V_2(LHS) = a$ thus a = 1
  - $2 * p^b = (p + 2)^c + 1$ again if we take $c = b + 1$, we get $RHS > LHS$ (since $p > 2$)

Thus $b = c$. Now we have $2^a * p^c = (p + 2)^c + 1$

- Note that, RHS $\equiv 2^c + 1$ mod p and the LHS is a multiple of p^c.
- For c = 3 RHS $\equiv$ 9 mod p which means p must be a factor of 9 (i.e. p = 3). However $V_3(LHS) > V_3(RHS)$ thus there are no solutions.
- Similarly for c = 4 RHS $\equiv$ 17 mod p -¿ p = 17 and for c = 5, p can take the values 3 and 11. In all of these cases $V_3(LHS) \not\equiv V_3(RHS)$
- For c = 1, p = 3. $2^a * 3 = 6$ Thus (1,1,1,3) is a solution.
- Similarly for c = 2, p = 5, $2^a * 25 = 50$ and (1,2,2,5) is a solution.

Hence overall there are **2** solutions.

§ Pentaweek 2 (05-10 Dec)

Day 1 | Evaluate the sum, $f(\mathrm{x}) = \sum_{k=0}^{\infty} \left\lfloor \frac{2^k+x}{2^{k+1}} \right\rfloor$]. where ($\lfloor x \rfloor$) denotes the Greatest integer Function.
Let $f(50.2)= n$. When 100 coins are tossed, what is the probability that exactly n are heads? Round your answer to two decimal places.

Solution:

By Hermite's identity, for real numbers x,

$$\lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor$$

Hence our sum telescopes

$$\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = \sum_{k=0}^{\infty} \left\lfloor \frac{n}{2^{k+1}} + \frac{1}{2} \right\rfloor = \sum_{k=0}^{\infty} \left(\left\lfloor \frac{n}{2^k} \right\rfloor - \left\lfloor \frac{n}{2^{k+1}} \right\rfloor \right) = \lfloor n \rfloor.$$

So $f(50.2)$=50

Probability of getting exactly 50 heads in tossing 100 coins is

$$\binom{100}{50} \left(\frac{1}{2}\right)^{100} = 0.08$$

Day 2 | Let

$$f(n) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{k} \times \binom{n-k}{\lfloor \frac{m-k}{2} \rfloor} \times 2^k$$

where $\lfloor x \rfloor$ denotes the Greatest integer Function and $\binom{n}{r}$ denotes the number of ways to choose r items from a set of n distinct items. Find the value of

$$\sum_{n=0}^{\infty} \frac{1}{f(n)}$$

Round your answer to two decimal places.

Solution:

Let us analyze the term in the summation step by step. Consider the term $\binom{n}{k} \times 2^k$. This represents the number of ways to distribute k coins to k people in the following scenario:

Out of n rooms, we select k rooms. The number of ways to do this is given by $\binom{n}{k}$.Each selected room contains 2 persons, and there are 2 possible ways to give a coin to one of them. Therefore, for k selected rooms, the total number of ways to distribute k coins is 2^k. Therefore, the term $\binom{n}{k} \times 2^k$ represents the number of ways to distribute k coins to k people in the selected rooms. k.

Now, consider the next term, $\binom{n-k}{\lfloor \frac{m-k}{2} \rfloor}$. After choosing k rooms, there are $n-k$ rooms remaining and there are $m-k$ coins left to distribute. Of the $n-k$ rooms, we select $\lfloor \frac{m-k}{2} \rfloor$ rooms, where each selected room receives 2 coins (one for each person in the room).

If $m-k$ is odd, one coin will remain after distributing coins in pairs to $\lfloor \frac{m-k}{2} \rfloor$ rooms. **This single leftover coin can be thought of as giving to a single person**.

To model this situation, we consider that there are $2n+1$ persons:

- 2 persons in each of the n rooms.
- 1 additional person who receives the leftover coin when $m-k$ is odd.

So the first summation that varies the term from k=0 to k=m will result in $\binom{2n+1}{m}$. As the summation is nothing but distributing m coins to $2n+1$ persons.

So second summation is nothing but $\sum_{m=0}^{n} \binom{2n+1}{m} = 2^{2n}$. Therefore

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

Day 3 | There is a certain series whose 1st term, s_0, is $\frac{5}{2}$, and the i'th term s_i, is given by $s_i = s_{i-1}^2 - 2$ for $i \geq 1$. Compute,

$$\prod_{i=0}^{\infty} \left(1 - \frac{1}{s_i}\right)$$

Solution:

Using the identity

$$a_{k+1} + 1 = (a_k - 1)(a_k + 1),$$

one checks by induction on n that

$$\prod_{k=0}^{n} \left(1 - \frac{1}{a_k}\right) = \frac{2}{7} \frac{a_{n+1} + 1}{a_0 a_1 \cdots a_n}.$$

Using the identity

$$a_{n+2}^2 - 4 = a_{n+1}^4 - 4a_{n+1}^2,$$

one also checks by induction on n that

$$a_0 a_1 \cdots a_n = \frac{2}{3}\sqrt{a_{n+1}^2 - 4}.$$

Hence

$$\prod_{k=0}^{n} \left(1 - \frac{1}{a_k}\right) = \frac{3}{7} \frac{a_{n+1} + 1}{\sqrt{a_{n+1}^2 - 4}}$$

tends to $\frac{3}{7}$ as a_{n+1} tends to infinity, hence as n tends to infinity.

Day 4 | Consider the Euclidean space $X = \mathbb{R}^n$ equipped with the metric (parameterized with the parameter $p \neq 0$)

$$d_p(x, y) = \left(\frac{(x_1 - y_1)^p + (x_2 - y_2)^p + ... + (x_n - y_n)^p}{n} \right)^{\frac{1}{p}}$$

First, ensure that this is a valid metric for all $p \neq 0$. If it is not, answer -1. If it is, then assume $n = 2$. If

$$s = \sup_{y=0, x_1=1, x_2>0} \{p : d_p(x, y) \text{ is convex in variable } p\}$$

then find $-2s$. 'Convex in p' means fix $x = (1, x_2)$ and $y = (0, 0)$ and examine the behaviour of the metric as p changes.
(Hint: plug $p = 1$, $p = -1$ and $p \to 0$ and see if the form of the metric seems familiar.)

References:

- Metric spaces
- Convexity
- Infimum and supremum

Solution:

The function given above has a special name. It is called the Power Mean (a generalized version of the Arithmetic, Geometric and Harmonic Means). The above is not a valid metric because $x_i - y_i$ could be negative, in which case the power mean is undefined. Hence the answer is -1. Even if you didn't catch this, $s = 0.5$, making $-2s = 1$. Search for journal articles related to 'convexity of the power mean'.

Day 5 | Let $\psi(k)$ denote the smallest positive integer such that for every $n \geq \psi(k)$ $(n, k \in \mathbb{Z})$ there always exists an integer of the form p^4 $(p \in \mathbb{Z})$ in the range $\big(n, k^2 n\big]$. Find the value of

$$\left(\sum_{k=2}^{2024} \psi(k)\right) - 20$$

Solution:

An interesting observation to make is that as the value of k increases, the size of the range $(n, k^2 n]$ also increases. You can assert with some confidence that after a sufficiently large k every possible range in n would include an integer of the form p^4. Let us try to see if this is true.

We want a p to exist such that $n < p^4 \leq k^2 n$, that is, $n^{1/4} < p \leq \sqrt{k} n^{1/4}$. A sufficient condition for an integer p to exist in this range would be $(\sqrt{k} n^{1/4} - n^{1/4}) \geq 1$. This condition always holds for any n when $k \geq 4$ because $(\sqrt{k} n^{1/4} - n^{1/4}) \geq (\sqrt{4} n^{1/4} - n^{1/4}) = n^{1/4} \geq 1$. So $\psi(k) = 1$ for all $k \geq 4$.

Using a similar approach, we can deduce that $\psi(k)$ can be at most 34 for $k = 2$ and at most 4 for $k = 3$. Manually checking till the upper bounds, we find that $\psi(2) = 21$ and $\psi(3) = 2$.

So, $\sum_{k=2}^{2024} \psi(k) = 21 + 2 + (2024 - 4 + 1) \times 1 = 2044$ and the final answer is $2044 - 20 = 2024$

§ Pentaweek 3 (10-15 Dec)

Day 1 | Let x, y, z, w be positive real numbers satisfying

$$(x+z)(y+w) = xz + yw$$

Find the smallest possible value of

$$P = \frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x}$$

Solution:

Using the AM-GM inequality

$$\left(\frac{x}{y} + \frac{z}{w}\right) + \left(\frac{y}{z} + \frac{w}{x}\right) \geq 2\sqrt{\frac{xz}{yw}} + 2\sqrt{\frac{wy}{xz}}$$

$$= 2\frac{xz + yw}{\sqrt{xyzw}} = 2\frac{(x+z)(y+w)}{\sqrt{xyzw}} \quad \text{(using the equation given)}$$

$$\geq 2\frac{(2\sqrt{xz})(2\sqrt{yw})}{\sqrt{xyzw}} = 8$$

Day 2 | Let S be the set of ordered pairs (x, y) such that

$$0 < x \leq 1, \quad 0 < y \leq 1,$$

and

$$\left\lfloor \log_2\left(\frac{1}{x}\right)\right\rfloor \quad \text{and} \quad \left\lfloor \log_5\left(\frac{1}{y}\right)\right\rfloor$$

are both even. Given that the area of the graph of S is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

The notation $\lfloor z \rfloor$ denotes the greatest integer that is less than or equal to z.

Solution:

$\left\lfloor \log_2\left(\frac{1}{x}\right)\right\rfloor$ is even when $x \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{8}, \frac{1}{4}\right) \cup \left(\frac{1}{32}, \frac{1}{16}\right) \cup \ldots$ and $\left\lfloor \log_5\left(\frac{1}{y}\right)\right\rfloor$ is even when $y \in \left(\frac{1}{5}, 1\right) \cup \left(\frac{1}{125}, \frac{1}{25}\right) \cup \left(\frac{1}{3125}, \frac{1}{625}\right) \cup \ldots$

The graph S constitutes of several rectangles, consisting of the intersection of x-strips and y-strips as found above. The width of the x-strips follow the sequence $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$ and the y-strips follow the sequence $\frac{4}{5}, \frac{4}{125}, \frac{4}{3125}, \ldots$. Thus, the area of the graph S is:

$$\left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots\right)\left(\frac{4}{5} + \frac{4}{125} + \frac{4}{3125} + \ldots\right) = \left(\frac{1/2}{1 - 1/4}\right)\left(\frac{4/5}{1 - 1/25}\right) = \frac{5}{9}$$

We have $m + n = 5 + 9 = 14$

Day 3 | For a positive integer N , let $f_N(x)$ be the function defined by

$$f_N(x) = \sum_{n=0}^{N} \frac{N + \frac{1}{2} - n}{(N+1)(2n+1)} \sin((2n+1)x).$$

Determine the smallest constant M such that $f_N(x) \leq M$ for all N and all real x.(Round your answer to two decimal place

Solution:

Left as an exercise to our fellow DCs.

Day 4 | Let $d : \mathbb{N} \to \mathbb{N}$ be defined as follows: If $n = 2^k a_k + 2^{k-1} a_{k-1} + \cdots + 2^0 a_0$, where $a_i \in \{0, 1\}$, then, $d(n) = \sum a_i$.

Let,

$$S = \sum_{k=1}^{2024} (-1)^{d(k)} k^3$$

Let, $P = S \mod 2024$, $P = uvwt$ (where u, v, w, t are the digits of the number P).

Two players each roll two standard dice, first player A, then player B. If player A rolls a sum of t, they win. If player B rolls a sum of w, they win. They take turns, back and forth, until someone wins. What is the probability that player A wins? (Round your answer to two decimal places.)

Solution:

Left as another exercise to our fellow DCs.

Day 5 | Consider ΔABC right-angled at A with $BC = 5$. Let BC be divided into 7 segments of equal length. Let the segment containing the midpoint of BC be l.
Find $\frac{\tan \alpha}{h}$ where α is the angle subtended by l at A and h is the altitude from A.

Solution:

Using coordinates, let A = (0,0), B = (b,0),and C = (0,c).Also,let PQ be the segment that contains the midpoint of the hypotenuse with P closer to B. Let n be number of parts in which it is divided equally, and let a be the hypotenuse BC length. Then,

$$P = \frac{n+1}{2} B + \frac{n-1}{2} C = \left(\frac{n+1}{2} b, \frac{n-1}{2} c \right), \quad Q = \frac{n-1}{2} B + \frac{n+1}{2} C = \left(\frac{n-1}{2} b, \frac{n+1}{2} c \right).$$

So,

$$\text{slope}(PA) = \tan(\angle PAB) = \frac{c}{b} \cdot \frac{n-1}{n+1}, \quad \text{slope}(QA) = \tan(\angle QAB) = \frac{c}{b} \cdot \frac{n+1}{n-1}.$$

Thus,

$$\tan(\alpha) = \tan(\angle QAB - \angle PAB) = \frac{\left(\frac{c}{b} \cdot \frac{n+1}{n-1} \right) - \left(\frac{c}{b} \cdot \frac{n-1}{n+1} \right)}{1 + \left(\frac{c}{b} \cdot \frac{n+1}{n-1} \right) \cdot \left(\frac{c}{b} \cdot \frac{n-1}{n+1} \right)}.$$

Simplifying the numerator and denominator:

$$= \frac{\frac{c}{b} \cdot \frac{4n}{n^2-1}}{1 + \frac{c^2}{b^2}} = \frac{4nbc}{(n^2-1)(b^2+c^2)} = \frac{4nbc}{(n^2-1)a^2}.$$

Since

$$[ABC] = \frac{1}{2}bc = \frac{1}{2}ah, \quad bc = ah,$$

we have

$$\tan(\alpha) = \frac{4nh}{(n^2-1)a},$$

as desired. Now, $n = 7$ and $a = 5$. After solving, we get

$$\frac{\tan(\alpha)}{h} = \frac{28}{240}.$$

§ Pentaweek 4 (15-20 Dec)

Day 1 | Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1 \le i,j \le n}$ be the matrix with entries defined as:

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \ge 0, \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose B is an $n \times n$ matrix with entries 0 or 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Determine the number of matrices B satisfying the above condition for any set of random numbers.

Solution:

Solution. Let $\mathbf{A} = (a_{ij})_{1 \le i,j \le n}$. Define

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i + y_j)(a_{ij} - b_{ij}).$$

On one hand, we have

$$S = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} b_{ij} \right) + \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} b_{ij} \right) = 0.$$

On the other hand, if $x_i + y_j \ge 0$, then $a_{ij} = 1$, which implies $a_{ij} - b_{ij} \ge 0$; if $x_i + y_j < 0$, then $a_{ij} = 0$, which implies $a_{ij} - b_{ij} \le 0$. Hence, $(x_i + y_j)(a_{ij} - b_{ij}) \ge 0$ for all i, j. Since $S = 0$, all $(x_i + y_j)(a_{ij} - b_{ij}) = 0$.

In particular, if $a_{ij} = 0$, then $x_i + y_j < 0$ and so $b_{ij} = 0$. Since a_{ij}, b_{ij} are 0 or 1, so $a_{ij} = b_{ij}$ for all i, j. Finally, since the sum of the elements in each row and each column of $\mathbf{B}$ is equal to the corresponding sum for the matrix $\mathbf{A}$, so $a_{ij} = b_{ij}$ for all i, j.

Day 2 | PV, KK, KV, and PJ play a game of tag. The game begins with PV being 'it' (PV is tagged initially), and each time some other player is tagged (Self-tags are not allowed). What are the number of distinct ways PV is 'it' again after 6 tags? (Note : PV can be 'it' in the middle also)

Solution:

Let x_n be the number of ways in which PV becomes 'it' again after n passes. Let y_n be the number of ways in which someone else becomes 'it' again other than PV after n passes. Then

$$x_n = 3y_{n-1},$$

and

$$y_n = x_{n-1} + 2y_{n-1}.$$

We also have $x_1 = 0$, $x_2 = 3$, $y_1 = 1$, and $y_2 = 2$.

Eliminating y_n and y_{n-1}, we get

$$x_{n+1} = 3x_n + 2x_{n-1}.$$

Thus

$$\begin{aligned}
x_3 &= 3x_2 + 2x_1 = 2 \times 3 = 6;\\
x_4 &= 3x_3 + 2x_2 = (3 \times 3) + (2 \times 6) = 9 + 12 = 21;\\
x_5 &= 3x_4 + 2x_3 = (3 \times 6) + (2 \times 21) = 18 + 42 = 60;\\
x_6 &= 3x_4 + 2x_5 = (3 \times 21) + (2 \times 60) = 63 + 120 = 183.
\end{aligned}$$

Day 3 | I have two strings x and y. String x is formed by using the letters of the word Atreya while y is formed by using the letters of the word Shivanshu. Let $n(i)$ denote the sum of the number of possibilities for x and y, when both are of length i. Find the sum of all possible values of $\gcd(n(j), n(k))$, where j and k are relatively prime.

Solution:

This question was infact not set by Shivanshu.

Day 4 | Given $0 \le x_1 \le x_2 \le \cdots \le x_n$,

$$\limsup_{n\to\infty} \left(\frac{\sum_{i=1}^{n} \sqrt{\frac{\sum_{k=i}^{n} x_k^2}{i}}}{\sqrt{n}\sum_{j=1}^{n} x_j} \right) = ?$$

Solution:

- We take $\sum_{j=1}^{n} x_j$ which is in the denominator, into the root.
- The purpose of this is to normalize x_i, so that it is in the range $[0, 1]$.
- Then we can change the variables as shown below.

$$\limsup_{n\to\infty} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{\frac{\sum_{k=i}^{n} \left(\frac{x_k}{\sum_{j=1}^{n} x_j}\right)^2}{i}} \right)$$

Let us change the variables now,

- $\frac{x_k}{\sum_{j=1}^{n} x_j} = v_i$
- $\sum_{i=1} n v_i = 1$
- $0 \le v_1 \le v_2 \le \cdots \le v_n \le 1$

Now we get the expression,

$$\limsup_{n\to\infty} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{\frac{\sum_{k=i}^{n} v_k^2}{i}} \right)$$

$\sum_{k=i}^{n} v_i^2$ has the maximum value $= 1$, when $v_n = 1$ and $v_j = 0 \; \forall \; j \neq n$, therefore,

$$\limsup_{n\to\infty} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{\frac{\sum_{k=i}^{n} v_k^2}{i}} \right) = \sum_{i=1}^{n} \frac{1}{\sqrt{ni}} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{\sqrt{\frac{i}{n}}} \right) \left(\frac{1}{n} \right) = \int_0^1 \frac{1}{\sqrt{x}} dx = 2$$

The answer is 2.

Day 5 | touching S_1, S_2 and C_0. In general, let C_n be the circle touching S_1, S_2 and C_{n-1} . Suppose that h_n is the distance of the centre of C_n from the x-axis and d_n is the diameter of C_n . Compute

$$\frac{h_{100}}{d_{100}}$$

Solution:

Theorem 0.0.1 (Shoemaker's Knife)

Let A, B, C be three collinear points (in that order) and construct three semicircles Γ_{AC}, Γ_{AB}, ω_0, on the same side of $\overline{AC}$, with diameters $\overline{AC}$, $\overline{AB}$, $\overline{BC}$, respectively. For each positive integer k, let ω_k be the circle tangent to Γ_{AC} and Γ_{AB} as well as ω_{k-1}.

For any positive integer n, the distance from the center of ω_n to $\overline{AC}$ is n times its diameter.

Proof. We prove this using inversion geometry.

Step 1: Inversion Setup Let's invert about point A with radius r chosen such that ω_n is orthogonal to our circle of inversion. This has the following effects:

- ω_n remains fixed by construction
- The semicircles Γ_{AB} and Γ_{AC} become parallel lines Γ^*_{AB} and Γ^*_{AC}
- Point A maps to infinity

Step 2: Properties After Inversion In the inverted image:

- Let d be the distance between the parallel lines
- All circles ω_k are tangent to both parallel lines
- Each ω_k is tangent to ω_{k-1}
- All circles in the chain become congruent with diameter d

Step 3: Key Observation Consider the center of ω_n. Let h be its distance from $\overline{AC}$. Due to the properties of inversion and the given conditions:

$$h = nd$$

where d is the diameter of ω_n.

Step 4: Conclusion When we invert back to the original configuration:

- The ratio of distances is preserved under inversion
- Therefore, in the original configuration, the distance from the center of ω_n to $\overline{AC}$ is n times its diameter

□

§ Pentaweek 5 (20-25 Dec)

Day 1 | Let $a_1 = 1$, $a_n = \sum_{k=1}^{n-1}(n-k)a_k$, $\forall\, n > 1$. Find a_{100}.
(Note: Some internet use (or a good calculator) might be required for the exact value.)

Solution:

Left as the final exercise to our fellow DCs?

Day 2 | The roots of a 6^{th} degree polynomial (of the form $x + iy$) lie on the ellipse $\frac{x^2}{2} + \frac{y^2}{3} = 1$ such that they divide the ellipse into 6 regions of equal area. One of the roots is given to be $x = \sqrt{2}$. Find the product of roots.
(Note: Some internet use (or a good calculator) might be required for the exact value.)

Solution:

Hint: An ellipse is a circle in a scaled up coordinate system.
If we define a new coordinate system where $X = \sqrt{2}x$ and $Y = \sqrt{3}y$, in this new coordinate system our ellipse can be represented by $X^2 + Y^2 = 1$. Since one of the roots lies on $X = 1, Y = 0$ and the roots subtend equal areas, we can think of them as the sixth roots of unity (in this new coordinate system). We can perform a transformation and then obtain the product of roots. [For example, $\frac{1+\sqrt{3}i}{2}$ in (X, Y) goes to $\frac{\sqrt{2}+3i}{2}$ in (x, y)]
Final Answer is: $\frac{-121}{8}$

Day 3 | Let A and E be opposite vertices of a regular octagon. A frog starts jumping from the vertex A. From any vertex except E of the octagon, it can jump to either of the two adjacent vertices. When it reaches the vertex E, the frog stops and stays there. Let e_n be the number of paths of exactly n jumps that start at A and end at E. Find

$$\lim_{n\to\infty} \frac{e_{2n+2}}{e_{2n}}$$

Solution:

Let's define $f(X, y)$ as the number of ways to reach vertex X in y moves.

$$\begin{aligned}
f(E, 2n) &= f(F, 2n-1) + f(D, 2n-1) \\
&= f(C, 2n-2) + f(G, 2n-2) \\
&= f(D, 2n-3) + f(B, 2n-3) + f(F, 2n-3) + f(H, 2n-3) \\
&= 2f(D, 2n-5) + 2f(F, 2n-5) + 4f(H, 2n-5) + 4f(B, 2n-5) \\
&= 4\left[f(B, 2n-5) + f(D, 2n-5) + f(F, 2n-5) + f(H, 2n-5)\right] \\
&\quad - 2\left[f(F, 2n-5) + f(D, 2n-5)\right] \\
&= 4f(E, 2n-2) - 2f(E, 2n-4)
\end{aligned}$$

By using the characteristic equation $x^2 - 4x + 2 = 0$, We get $2 + \sqrt{2}$ and $2 - \sqrt{2}$ as the roots which implies $e_{2n} = \frac{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}}{\sqrt{2}}$
Therefore $\lim_{n\to\infty} \frac{e_{2n+2}}{e_{2n}} = \lim_{n\to\infty} \frac{(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}}{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}} = 2 + \sqrt{2}$

The final answer is 3.41

Day 4 |

$$f(n) = \begin{cases} n-2, & \text{for } n > 3000; \\ f(f(n+5)), & \text{for } n \leq 3000. \end{cases}$$

Find the value of $f(2024)$

Solution:

We note that the recursive function $f(n)$ will always return one of the values $2999, 3000$, or 3001 for all $n < 3000$. Let's analyze the behavior of the function in detail for $n = 2024$:

$$f(2024) = f(f(2029)) = f(f(f(2034))) = \cdots = f^{197}(3004).$$

Now, using the definition of the function:

$$f^{197}(3004) = f^{196}(3002), \quad f^{196}(3002) = f^{195}(3000), \quad f^{195}(3000) = f^{194}(3001),$$

$$f^{194}(3001) = f^{193}(2999), \quad f^{193}(2999) = f^{192}(3000).$$

As we can see, we have returned to the same argument 3000, which implies that the function enters a loop of size 3 for arguments $2999, 3000$, and 3001.

To determine the final result, we calculate the residue of 193 modulo 3:

$$193 \div 3 = 64 \,(\text{full cycles}) \quad \text{and residue } 1.$$

Thus:

$$f^{193}(2999) = f(2999) = f(f(3004)) = f(3002) = 2999.$$

Final Answer is : $f(2024) = 2999.$

Day 5 | Let, $ab = \pi y$ Find the sum of first 6 significant figures of y.

$$a = \lim_{n\to\infty} \sum_{k=2}^{n} \ln\left(\frac{2^k}{2^k - 1}\right) \prod_{p|k} \frac{p-1}{p},$$

where the product is taken over all prime divisors p of k.

$$b = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{2^k}{2^k - 1}\right) \sum_{d|k} \frac{f(d)}{d},$$

where the summation is taken over all divisors d of k,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a square-free positive integer with an even number of prime factors,} \\ -1 & \text{if } x \text{ is a square-free positive integer with an odd number of prime factors,} \\ 0 & \text{if } x \text{ has a squared prime factor.} \end{cases}$$

A square-free positive integer is a positive integer that is not divisible by any perfect square other than

1

Solution:

$$\sum_{d|k} \frac{f(d)}{d} = \frac{\varphi(k)}{k}$$

$$\prod_{p|k} \frac{p-1}{p} = \frac{\varphi(k)}{k}$$

where $\varphi(k)$ is euler's totient function

$$\ln\left(\frac{2^k}{2^k-1}\right) = -\ln\left(\frac{1}{1-2^{-k}}\right) = \sum_{m=1}^{\infty} \frac{2^{-km}}{m}$$

substituting the above expressions in a and b we get,

$$a = \left(\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{2^{-km}}{km}\varphi(k)\right) - \ln 2$$

(we have subtracted $\ln 2$ as the the summation in a starts from $k=2$ and **NOT** from $k=1$)

$$b = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{2^{-km}}{km}\varphi(k)$$

Now we make the observation that both k and m take all positive integer values.

- Thus, $km = s$, will occur multiple times, where s is a positive integer.
- km will be equal to s whenever k and m are factors of s
- So let us convert this to a single summation in s.
- When we convert this to a single summation in s, we observe that each term will be multiplied by $\sum_{d|s} \varphi(d)$, where d represents all the factors of s, **as shown below**.

$$b = \sum_{s=1}^{\infty}\left(\frac{2^{-s}}{s}\sum_{d|s}\varphi(d)\right)$$

Now we substitute the identity $\sum_{d|s} \varphi(d) = s$ in the above expression for b. We get,

$$b = \sum_{s=1}^{\infty} 2^{-s} = 1$$

and

$$a = 1 - \ln 2$$

it is given that,

$$ab = \pi y$$

$$y = \frac{1-\ln 2}{\pi} \approx 0.097674286$$

Sum of the first 6 significant figures is $= 35$