Mathematics Guild
Problem Set - 5

Challenge posed on: 22/07/2025 **Challenge to be conquered by: 28/07/2025**

1 Overview

- **Topics focused:**
  - Calculus
- **Challengers:**
  - Balasathya Saravanan
  - Laxmi Narasimha Reddy
  - Matheshwaran S
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Happy solving :)

2 Problems

1. **Real answers live in imaginary places?** **(3 points)** Sometimes imagination is better than reality.

$$\int_0^1 tan^{-1}\frac{\sqrt{3}(1-x)}{1+x} \cdot \ln(1-x+x^2)\,dx$$

2. **Ah! I've done these during JEE!** **(3 points)** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that:

$$2f(x) + 2f\left(1-\frac{1}{x}\right) = 3\tan^{-1}x \quad \text{for all real } x \text{ (with } x \neq 0).$$

Return the value of

$$\int_0^1 f(x)\,dx$$

3. **Not Another One, Mr. Feynman!** "If you can't solve a big problem, break it into little pieces and figure out what each one is doing." -Feynman probably

$$I(m,n) = \int_0^\infty \frac{\ln(1+x^m)}{1+x^n}\,dx$$

Find:

- $I(2,4)$ **(2 points)**
- $I(4,2)$ **(2 points)**
- $I(4,4)$ **(2 points)**

4. **Well... idea gone, carry on!** **(3 points)** Consider the differential equation

$$y'(x) = 2y(x) + y(-x),$$

with the initial condition $y(0) = 1$. Show that the limit

$$L = \lim_{x\to\infty} \left(y(x) e^{-\sqrt{3}x} \right)$$

exists, and find its value.

5. **A Feynman integral with a complex twist!** **(3 points)**

$$\int_0^1 \frac{\sin(\ln x) \prod_{k=0}^{n-1} \cos(2^k \ln x)}{\ln x}\, dx$$

6. **A long time ago, in a galaxy far, far away...** **(3 points)** Years after his duel with Master Obi-Wan Kenobi, Darth Vader (*he was not granted the rank of master*) returns to **Mustafar**. He seeks to quantify the stability of the permanent *Force Scar* his duel left upon the world. He determines that the scar's resistance to change—its spiritual inertia—can be calculated as the **Force Moment of Inertia** (a quantity that measures the distribution of dark side energy).

The **Dark Side Density** at a point $\mathbf{r} = \langle x, y \rangle$ on the plains is given by the function:

$$\rho(\mathbf{r}) = \exp\left(-\left[(\mathbf{r}\cdot\mathbf{v}_1)^2 + (\mathbf{r}\cdot\mathbf{v}_2)^2\right]\right)$$

where $\mathbf{v}_1 = \langle 2, 1 \rangle, \quad \mathbf{v}_2 = \langle 1, 2 \rangle$ are the *Axes of Fate* that defined the duel.

The **Force Moment of Inertia** J is calculated by integrating the density multiplied by the squared distance from the duel's nexus at the origin:

$$J = \iint_{\mathbb{R}^2} (x^2 + y^2)\, \rho(x, y)\, dx\, dy$$

Lord Vader orders his imperial officers to find the exact value of J or else they are gonna get choked. Help the imperials survive by calculating J! (The imperials are smart but not in math sadly.. hence they seek help from you)

7. **The Spiral Code** **(3 points)**

In the ruins of a forgotten dimension, a young prodigy named **Euler** was chosen to decode the heartbeat of the Spiral Gate a portal that twisted reality through oscillations and exponential surges.

Inscribed on the gate was a riddle encoded in motion:

$$\int e^{2\cos(2x)} \cdot \cos\left(2\sin(2x) + 4x + 8\right)\, dx$$

Help Euler to unlock the gate.

8. **Infinite Castle** **(3 points)**

In the dead of night, *Muzan Kibutsuji* envisioned the construction of the **Infinite Castle** a realm that folds and stretches space infinitely.

But to begin, he needed a strange kind of symmetry: a function that remains unchanged even when the input is warped beyond recognition.

A mysterious condition emerged:

$$f(x) = f(x^{2025}) \quad \text{for all } x \in \mathbb{R}.$$

The function $f : \mathbb{R} \to \mathbb{R}$ is continuous. Muzan believed such symmetry would lay the foundation of the castle.

Find all functions f to help Muzan build infinte castle.

9. **This looked hard at 3AM but I ain't sure man (3 points)**

Let $f : (0, \infty) \to \mathbb{R}$ be a continuously differentiable function. It is known that, for every pair of real numbers a,b , the point in $[a, b]$ guaranteed by the Lagrange Mean Value Theorem always occurs at the geometric mean $\sqrt{ab}$.

Determine all such functions f.