# Mathematics Guild
# Solutions to Problem Set - 4

**Challenge posed on: 15/07/2025** **Challenge conquered by: 21/07/2025**

## 1 Overview

- **Topics focused:**
  - Linear Algebra
  - Abstract Algebra
- **Challengers:**
  - Hafiz Rahman Elikkottil
  - Ayush Gangal
  - Srikar JV
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Happy solving :)

## 2 Problems

1. **The Odd Resultant Mystery** We give a proof by induction on $n$.

**Base Case** ($n = 1$)**:**

For $n = 1$, we have a single unit vector. The magnitude of the resultant is 1, satisfying the statement.

**Inductive Step:**

Assume the statement holds for some odd $n = 2k - 1$; that is, for any configuration of $2k - 1$ unit vectors on one side of the $x$-axis, the magnitude of their sum is at least 1.

We must show that it holds for $n = 2k + 1$ as well.

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{2k+1}$ be unit vectors in the plane, all lying in the same open half-plane defined by the $x$-axis. Let $\theta_i$ be the angle that $\vec{v}_i$ makes with the positive $x$-axis, with

$$0 < \theta_1 \le \theta_2 \le \cdots \le \theta_{2k+1} < \pi.$$

Consider the sum of the vectors:

$$\vec{S} = \vec{v}_2 + \vec{v}_3 + \cdots + \vec{v}_{2k}$$

By the induction hypothesis, $|\vec{S}| \ge 1$. Now, consider adding two more unit vectors $\vec{v}_1$ and $\vec{v}_{2k+1}$ (the remaining two vectors):

$$\vec{R} = \vec{S} + \vec{v}_1 + \vec{v}_{2k+1}$$

If we show $|\vec{R}| \ge 1$, then we are done.

Since $\vec{v}_1$ and $\vec{v}_{2k+1}$ are unit vectors, the direction of their sum $\vec{v}_1 + \vec{v}_{2k+1}$ will be along the angle $\frac{\theta_1+\theta_{2k+1}}{2}$ with $0 < \theta_1 \le \theta_{2k+1} < \pi$.

Let $\beta$ be the angle that $\vec{S}$ makes with the +ve $x$-axis ($\theta_1 \le \beta \le \theta_{2k+1}$). We can show that the angle between $\vec{S}$ and $\vec{v}_1 + \vec{v}_{2k+1}$ is at most $\frac{\pi}{2}$.

By the general fact that if the angle between two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is at most $\frac{\pi}{2}$, then

$$|\overrightarrow{u} + \overrightarrow{v}| \geq \max\{|\overrightarrow{u}|, |\overrightarrow{v}|\},$$

we have

$$\left|\vec{R}\right| = \left|\vec{S} + \vec{v}_1 + \vec{v}_{2k+1}\right| = \left|\vec{S} + (\vec{v}_1 + \vec{v}_{2k+1})\right| \geq \max\left\{|\vec{v}_1 + \vec{v}_{2k+1}|, \left|\vec{S}\right|\right\} \geq \left|\vec{S}\right| \geq 1,$$

as required.

Thus, for all odd $n$, the magnitude of the sum of $n$ unit vectors lying on the same side of the $x$-axis is at least 1.

2. **One small step on min, One giant fall on max** Let $v_1, \dots, v_n$ denote the rows of $A$. Perform the row operations:

$$v_n \to v_n - v_{n-1}, \quad v_{n-1} \to v_{n-1} - v_{n-2}, \ \dots, \ v_2 \to v_2 - v_1.$$

Since for each $k \geq 2$,

$$(v_k - v_{k-1})_k = \frac{1}{k} - \frac{1}{k-1},$$

the resulting matrix is upper-triangular with diagonal entries:

$$1, \quad \frac{1}{2} - 1, \quad \frac{1}{3} - \frac{1}{2}, \ \dots, \ \frac{1}{n} - \frac{1}{n-1}.$$

Hence,

$$\det A = \prod_{k=2}^{n}\Big(\frac{1}{k} - \frac{1}{k-1}\Big) = \prod_{k=2}^{n}\Big(-\frac{1}{k(k-1)}\Big) = (-1)^{n-1}\frac{1}{(n-1)!\,n!}.$$

Perform the column operations:

$$C_k \ \to \ C_k - C_{k+1}, \quad k = 1, 2, \dots, n-1$$

Then, for each $1 \leq k \leq n-1$,

$$(C_k)_i = \frac{1}{\max(i,k)} - \frac{1}{\max(i,k+1)},$$

which vanishes for all $i > k$. Thus, the new matrix is upper-triangular with diagonal entries:

$$d_k = \frac{1}{k} - \frac{1}{k+1} \quad (1 \leq k < n), \qquad d_n = \frac{1}{n}$$

Therefore,

$$\det A = \Big(\prod_{k=1}^{n-1}\big(\tfrac{1}{k} - \tfrac{1}{k+1}\big)\Big) \cdot \frac{1}{n} = \prod_{k=1}^{n-1}\frac{1}{k(k+1)} \cdot \frac{1}{n} = \frac{1}{(n!)^2}.$$

Note that exactly $2k-1$ pairs $(i,j)$ satisfy $\max(i,j) = k$. Therefore,

$$S_n = \sum_{k=1}^{n}(2k-1)\,\frac{1}{k} = 2\sum_{k=1}^{n}1 - \sum_{k=1}^{n}\frac{1}{k} = 2n - H_n,$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the $n$th harmonic number.

3. **The Grand Eigenvalue Challenge** Let the table be represented by a real symmetric $2025 \times 2025$ matrix $A = (a_{ij})$ satisfying $|a_{ij} - 2025| \leq 1$ for every $1 \leq i, j \leq 2025$. Denote the largest eigenvalue of A by $\lambda(A)$.

**Spectral Theorem for Real Symmetric Matrices-**

Let $A \in \mathbb{R}^{n \times n}$ be real and symmetric. Then:

(a) The eigenvalues of $A$ are real.

**Proof.** Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$, and let $x \in \mathbb{C}^n$ be a nonzero eigenvector such that $Ax = \lambda x$. Then

$$x^* Ax = x^*(\lambda x) = \lambda x^* x.$$

Since $A$ is symmetric, $x^* Ax = (Ax)^* x = (\lambda x)^* x = \overline{\lambda} x^* x$. Hence,

$$\lambda x^* x = \overline{\lambda} x^* x.$$

Since $x \neq 0$, we have $x^* x > 0$, so $\lambda = \overline{\lambda}$, i.e., $\lambda$ is real.

(b) A is orthogonally diagonalizable.

(c) There is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$.

**Theorem:** A matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable if and only if there exists a basis of $\mathbb{F}^{n \times 1}$ consisting of eigenvectors of $A$.

Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$, and let $\lambda_1$ and $\lambda_2$ be distinct eigenvalues of $A$ with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$, respectively. We aim to show that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

From the definition of eigenvectors and eigenvalues, we have:

$$A\vec{v}_1 = \lambda_1 \vec{v}_1, \quad A\vec{v}_2 = \lambda_2 \vec{v}_2.$$

Multiplying both sides of the first equation on the left by $\vec{v}_2^\top$, and both sides of the second equation on the left by $\vec{v}_1^\top$, we get:

$$\vec{v}_2^\top A\vec{v}_1 = \lambda_1 \vec{v}_2^\top \vec{v}_1, \quad \vec{v}_1^\top A\vec{v}_2 = \lambda_2 \vec{v}_1^\top \vec{v}_2.$$

Note that the quantity $\vec{v}_1^\top A\vec{v}_2$ is a scalar, so:

$$\vec{v}_1^\top A\vec{v}_2 = (\vec{v}_1^\top A\vec{v}_2)^\top = \vec{v}_2^\top A^\top \vec{v}_1.$$

Since $A$ is symmetric, $A^\top = A$, and thus:

$$\vec{v}_1^\top A\vec{v}_2 = \vec{v}_2^\top A\vec{v}_1.$$

Therefore, we can equate the right-hand sides of the two earlier equations:

$$\lambda_1 \vec{v}_2^\top \vec{v}_1 = \lambda_2 \vec{v}_1^\top \vec{v}_2.$$

But $\vec{v}_2^\top \vec{v}_1 = \vec{v}_1^\top \vec{v}_2$, so we get:

$$\lambda_1 \vec{v}_1^\top \vec{v}_2 = \lambda_2 \vec{v}_1^\top \vec{v}_2.$$

Since $\lambda_1 \neq \lambda_2$, it follows that:

$$\vec{v}_1^\top \vec{v}_2 = 0,$$

which shows that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

The set of $n$ eigenvectors can be chosen to be orthogonal (as shown above, at least across distinct eigenspaces).

For eigenvectors corresponding to the same eigenvalue (i.e., within the same eigenspace), we can apply the Gram–Schmidt process to obtain an orthonormal basis within each eigenspace.

Therefore, the entire space $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of the matrix.

Now, let $\mathbf{1} = \sqrt{\frac{1}{2025}}[1, 1, \ldots, 1]^T \in \mathbb{R}^{2025}$. The scaling is just so that $\langle \mathbf{1}, \mathbf{1} \rangle = 1$.

Let $n = 2025$

From (c), we can say that any vector $x \in \mathbb{R}^n$ can be written as a linear combination of the eigenvectors of a real symmetric matrix $A$:

$$x = \sum_{i=1}^{n} \alpha_i v_i, \quad \text{where } \alpha_i = \langle x, v_i \rangle = v_i^T x.$$

Let's apply this to the specific vector:

$$\mathbf{1} = \sum_{i=1}^{n} \alpha_i v_i.$$

Since the eigenvectors $\{v_i\}_{i=1}^n$ form an orthonormal basis, we can compute each coefficient using the inner product:

$$\alpha_i = \langle \mathbf{1}, v_i \rangle = v_i^T \mathbf{1}.$$

Now consider the squared norm of $\mathbf{1}$. For any $x \in \mathbb{R}^n$, we have:

$$\|x\|^2 = \langle x, x \rangle.$$

So in our case:

$$\|\mathbf{1}\|^2 = \left\langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{n} \alpha_j v_j \right\rangle.$$

On simplifying it,

$$\|\mathbf{1}\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \langle v_i, v_j \rangle.$$

Since the $v_i$'s are orthonormal, we have:

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Therefore, the double sum simplifies to:

$$\|\mathbf{1}\|^2 = \sum_{i=1}^{n} \alpha_i^2 = 1$$

Now, we can write

$$\frac{\sum_{1 \le i,j \le n} a_{ij}}{2025} = \langle A\mathbf{1}, \mathbf{1} \rangle = \mathbf{1}^T A \mathbf{1}$$

Let's compute the expression $\mathbf{1}^T A \mathbf{1}$.

From the eigenvector decomposition, we can write:

$$\mathbf{1} = \sum_{i=1}^{n} \alpha_i v_i, \quad \text{where each } \alpha_i = v_i^T \mathbf{1}.$$

Then,

$$A\mathbf{1} = A\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i A v_i = \sum_{i=1}^{n} \alpha_i \lambda_i v_i,$$

Now we compute:

$$\mathbf{1}^T A\mathbf{1} = \left(\sum_{i=1}^{n} \alpha_i v_i\right)^T \left(\sum_{j=1}^{n} \alpha_j \lambda_j v_j\right).$$

On simplifying it,

$$\mathbf{1}^T A\mathbf{1} = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \lambda_j (v_i^T v_j).$$

Since the eigenvectors $\{v_i\}$ form an orthonormal basis, we have:

$$v_i^T v_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

So the double sum reduces to:

$$\mathbf{1}^T A\mathbf{1} = \sum_{i=1}^{n} \alpha_i^2 \lambda_i.$$

Now, as $\lambda_i \leq \lambda(A)$, we can write:

$$\sum_{i=1}^{n} \alpha_i^2 \lambda_i \leq \sum_{i=1}^{n} \alpha_i^2 \lambda(A) = \lambda(A) \sum_{i=1}^{n} \alpha_i^2 = \lambda(A)$$

Thus,

$$\frac{\sum_{1\leq i,j\leq n} a_{ij}}{2025} = \langle A\mathbf{1}, \mathbf{1}\rangle \leq \lambda(A).$$

As, $2024 \times 2025 \leq \frac{\sum_{1\leq i,j\leq n} a_{ij}}{2025}$, we have $\lambda(A) \geq 2025 \times 2024$

Thus,

$$\boxed{\lambda(A) \geq 2025 \times 2024}$$

Let there be a unit eigenvector $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \lambda(A)\cdot\mathbf{x}, \quad \text{and} \quad \|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x} = 1$$

Therefore,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(\lambda(A)\cdot\mathbf{x}) = \lambda(A)\cdot\mathbf{x}^T\mathbf{x} = \lambda(A).$$

Expanding $\mathbf{x}^T A\mathbf{x}$ as $\sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j$, we can say:

$$\lambda(A) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j \leq 2026 \times \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j = 2026 \times (\sum_{i=1}^{n} x_i)^2$$

On applying Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n \cdot \sum_{i=1}^{n} x_i^2 = n \cdot \|x\|^2 = n.$$
$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n.$$

Thus,

$$\boxed{\lambda(A) \leq 2026 \times 2025}$$

Both of these bounds are achievable by setting the matrix as $2024J$ or $2026J$ respectively, where $J$ is the all-one matrix in $\mathbb{R}^{2025\times 2025}$ as we have eigenvector $\mathbf{1}$.

**Conclusion:**

(a) The highest possible value of the Grand Eigenvalue is $\boxed{2025 \times 2026 = 4{,}102{,}650}$.

(b) The lowest possible value of the Grand Eigenvalue is $\boxed{2025 \times 2024 = 4{,}098{,}600}$.

4. **Tired... here you go with the easy one** Let $G$ be a finite group under the operation $*$. Fix an element $a \in G$. Consider the sequence

$$a,\ a * a,\ a * a * a,\ \ldots,\ \underbrace{a * a * \cdots * a}_{k \text{ times}},\ \ldots$$

As $G$ is finite and the sequence above consists of elements in $G$, then, some elements in this sequence must repeat. That is, there exist integers $m < n$ such that

$$\underbrace{a * a * \cdots * a}_{n \text{ times}} = \underbrace{a * a * \cdots * a}_{m \text{ times}}$$

Define $b = \underbrace{a * a * \cdots * a}_{m \text{ times}}$, so the above reads:

$$b = b * \underbrace{a * a * \cdots * a}_{n-m \text{ times}}.$$

Multiply both sides on the left by the inverse of $b$ (since every element in a group has an inverse):

$$e = \underbrace{a * a * \cdots * a}_{n-m \text{ times}}.$$

Thus, for $k = n - m > 0$, we have $a$ composed with itself $k$ times equals the identity. Therefore, for every $a \in G$, there exists a positive integer $n$ such that

$$a^{*n} = \underbrace{a * a * \cdots * a}_{n \text{ times}} = e$$

(where $a^{*n}$ denotes $a$ operated with itself $n$ times).

5. **Ring-A-Ring-A-Roses, Pocket full of zeroes**

$$S \;=\; a_1 + a_2 + \cdots + a_n \;=\; 0$$

For each integer $k \geq 1$,

$$S^k = (a_1 + a_2 + \cdots + a_n)^k = 0$$

Since each $a_i^2 = a_i$, every monomial collapses either to a single $a_i$ or to a mixed product $a_i a_j$ $(i \neq j)$. Comparing coefficients yields:

$$\sum_{i=1}^{n} a_i^k = \sum_{i=1}^{n} a_i = 0, \quad \sum_{i \neq j} a_i a_j = 0, \quad \forall\, k \geq 1,$$

so all power sums $s_k = \sum_i a_i^k$ vanish.

Consider the monic polynomial in $D[t]$ with "roots" $a_1, \ldots, a_n$:

$$f(t) = \prod_{i=1}^{n} (t - a_i) = t^n - c_1\, t^{n-1} + c_2\, t^{n-2} - \cdots + (-1)^n c_n$$

Newton's identities

$$k\, c_k = \sum_{i=1}^{k} (-1)^{\,i-1}\, c_{\,k-i}\, s_i$$

imply inductively $c_1 = c_2 = \cdots = c_n = 0$, since each $s_i = 0$ and $k \neq 0$ in $D$. Hence, $f(t) = t^n$ in $D[t]$.

Fix any $j$. Dividing $t^n$ by the factor $(t - a_j)$ in $D[t]$ leaves a constant remainder $r_j$, but $f(t) = t^n$ is divisible by $(t - a_j)$, so $r_j = 0$. Substituting $t = a_j$ gives:

$$a_j^n = 0$$

Since $a_j$ is idempotent,

$$a_j = a_j^2 = a_j^3 = \cdots = a_j^n = 0$$

As $j$ was arbitrary, all $a_i = 0$.

6. **Escaping the matrix? (The determinant can't)** By Hadamard's inequality,

$$|\det C|^2 \leq \prod_{i=1}^{n} \|c_i\|^2 = \prod_{i=1}^{n} \big\|a_i + \varepsilon_i b_i\big\|^2 = \prod_{i=1}^{n} \big(\|a_i\|^2 + 2\varepsilon_i \langle a_i, b_i\rangle + \|b_i\|^2\big)$$

Setting $x_i = \langle a_i, b_i \rangle$, we get

$$|\det C|^2 \leq 2^n \prod_{i=1}^{n} (1 + \varepsilon_i x_i) = 2^n\, A(\varepsilon, x, n),$$

where

$$A(\varepsilon, x, n) = \prod_{i=1}^{n} (1 + \varepsilon_i x_i), \qquad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$$

We will show by induction on $n$ that

$$\sum_{\varepsilon_i = \pm 1} A(\varepsilon, x, n) = 2^n$$

For $n = 1$, $A(\varepsilon, x, 1) = 1 + \varepsilon_1 x_1$ and

$$\sum_{\varepsilon_1 = \pm 1} (1 + \varepsilon_1 x_1) = (1 + x_1) + (1 - x_1) = 2$$

Assume the claim for $n-1$. Then

$$\sum_{\varepsilon_i=\pm 1} A(\varepsilon,x,n) = \sum_{\varepsilon_n=\pm 1} (1+\varepsilon_n x_n) \sum_{\varepsilon_1,\ldots,\varepsilon_{n-1}} A(\varepsilon,x,n-1) = (1+x_n+1-x_n)\,2^{n-1} = 2^n$$

Since there are $2^n$ choices of $\varepsilon$, the average of $A(\varepsilon,x,n)$ is 1. So, for some choice $\varepsilon_1,\ldots,\varepsilon_n$, we have $A(\varepsilon,x,n)\le 1$. Hence,

$$|\det C|^2 \le 2^n \quad\Longrightarrow\quad |\det C| \le 2^{n/2}$$

7. **This one looks odd**

$$T_m = \binom{x_m-1}{2m-1}$$

$$\Rightarrow (2m-1)!\times T_m = (x+m-1)(x+m-2)\cdots(x-m+1) = x\prod_{i=1}^{m-1}(x^2-i^2)$$

So $T_m$ is an odd polynomial of degree 2m-1. Therefore, $T_1, T_2, \cdots, T_n$ span the same space as $x.x^3, ...x^{2n-1}$ and any odd polynomial of degree $\le 2n-1$ can be written as a linear combination of these terms.

8. **Invert it** Let $d(a,b)=1$ if $b|a$ else 0. Consider the vector $v(m)\ \epsilon\ R^n$.

$$v(m) = \begin{bmatrix} d(m,1) & d(m,2) & \cdots & d(m,n) \end{bmatrix}$$

We can see that the rows of $M(n)$ are $v(2), v(3), \cdots, v(n+1)$.

Our proof depnds on the following result. If the prime factorisation of $n+1$ is $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, then

$$\sum_I (-1)^{|I|} v\left(\frac{n+1}{\prod_{i\in I} p_i}\right) = (0,0,\ldots,0),$$

where we are summing over all subsets of $\{1,2,\ldots,k\}$ i.e., for all the numbers of n+1 that we get by dividing it with its square free factors. $|I|$ is the number of elements in $I$.

**Proof.** Let us consider the $j$th component of the left side of the equation, where $1\le j\le n$.

If $n+1$ is not divisible by $j$, then the $j$th component of each term is 0. Hence, the $j$th component of the left side of the equation is 0.

If $n+1$ is divisible by $j$, then we can write

$$\frac{n+1}{j} = p_1^{b_1}p_2^{b_2}\cdots p_k^{b_k},$$

where $0\le b_i\le a_i$ for each $i$.

Let $J=\{1\le i\le k \mid b_i>0\}$. Then the $j$th component of the left side of the equation is

$$\sum_I (-1)^{|I|} d\left(\frac{n+1}{\prod_{i\in I} p_i}, \frac{n+1}{p_1^{b_1}p_2^{b_2}\cdots p_k^{b_k}}\right) = \sum_I (-1)^{|I|} d\left(p_1^{b_1}p_2^{b_2}\cdots p_k^{b_k}, \prod_{i\in I} p_i\right)$$

$$= \sum_{I\subseteq J} (-1)^{|I|} d\left(p_1^{b_1}p_2^{b_2}\cdots p_k^{b_k}, \prod_{i\in I} p_i\right) = \sum_{I\subseteq J}(-1)^{|I|} = 0.$$

The first equality uses the fact that $\frac{n+1}{a}$ is divisible by $\frac{n+1}{b}$ if and only if $b$ is divisible by $a$.

The second and third equalities use the fact that $d\left(p_1^{b_1}p_2^{b_2}\cdots p_k^{b_k}, \prod_{i\in I} p_i\right)$ is equal to 1 if $I \subseteq J$, and equal to 0 otherwise.

The final equality uses the fact that $j \leq n$, so $J \neq \emptyset$.

We now consider the two cases:

**a)** n+1 is not square free:
This implies that some combination of vectors $v(2), v(3), \cdots, v(n+1)$ is equal to a zero vector. by performing the above summation as a row operation on the $n^{th}$ row, we get all zeroes in that row. Therefore the matrix is not invertible.

**b)** n+1 is square free: In this case, we get that some linear combination of $v(1), v(2), v(3), \cdots, v(n+1)$ is equal to the zero vector. The problem here is that $v(1)$ is not a row of the $M(n)$. So if we perform the above summation excluding the $v(1)$ term on the $n^{th}$ row, we get $\pm v(1)$ as the last row, where $v(1)$ is just $\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$. If we look at the cofactor matrix of the 1 in the last row, we can see that it is a lower triangular matrix with leading diagonal elements all being 1. Therefore $|M(n)| = \pm 1$ and therefore it is invertible.