Mathematics Guild
Problem Set - 4

Challenge posed on: 15/07/2025 **Challenge conquered by: 21/07/2025**

1 Overview

- **Topics focused:**
  - Linear Algebra
  - Abstract Algebra
- **Challengers:**
  - Hafiz Rahman Elikkottil
  - Ayush Gangal
  - Srikar JV
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Happy solving :)

2 Problems

1. **The Odd Resultant Mystery** (3 points) Ayush loves to tackle puzzles involving geometry and vector fields. One fine summer afternoon, he imagined a collection of n unit vectors (all lying in the xy plane), starting from the origin and pointing in various directions – but with one constraint: all vectors lie on the same side of the x axis. As he explored different configurations of these n vectors, he noticed a fascinating property. He observed that whenever n is odd, their resultant always had a magnitude greater than or equal to 1 regardless of the configuration in which the vectors are aligned. Can you help Ayush understand why this always holds true for odd n?

2. **One small step on min, One giant fall on max** (3 points) Bala searching through some previous olympiad archives finds an interesting problem in which one is asked to find $\det(A)$ for which A_{ij} is defined to be $\frac{1}{min(i,j)}$ where the shape of the square matrix A is n. He then wonders what the determinant would be if it were $A_{ij} = \frac{1}{max(i,j)}$.

   Unfortunately, Bala isn't able to solve this and he asks Harshath to solve it for him. Harshath solves the question in his mind and forgets to send the solution to Bala. SMMC is the next day and Bala had no choice but to put it up on the Math Guild Page. Can you help Bala ace SMMC by solving for both determinants in terms of n?

   And, while you are at it, find a closed form representation for the following summation:

   $$\sum_{1\leq i,j\leq n} \frac{1}{\max(i,j)}$$

   Express your answer in closed form in terms of the n^{th} harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

3. **The Grand Eigenvalue Challenge** (3 points) You are a mathematician in the ancient city of **Numeria**, where the wise council presents you with a mysterious challenge:

   They hand you a giant square table, with 2025 rows and 2025 columns. Each cell of the table contains a real number, and the table has a magical property:

- Every number in the table is very close to 2025 – in fact, each one differs from 2025 by at most 1.
- The table is perfectly symmetric: the number in row i, column j is always the same as the number in row j, column i.

The council tells you that the table hides a secret number, called the **Grand Eigenvalue** – the largest eigenvalue of the table.

The council presents you with the following questions:

a) What is the highest possible value this Grand Eigenvalue can take?
b) What is the lowest possible value it can take?

Can you unravel the mystery and impress the council with your answer?

4. **Tired... here you go with the easy one** (3 points) Consider a finite group G under a special operation $*$. Show that for any element $a \in G$, there exists a positive integer n, depending on a, such that

$$\underbrace{a * a * \cdots * a}_{n \text{ times}} = e$$

where e is the identity element of the group (for any element $x \in G$, $x * e = e * x = x$)

5. **Ring-A-Ring-A-Roses, Pocket full of zeroes** (3 points) Let D be a commutative ring of characteristic zero which is torsion-free as a $\mathbb{Z}$-module. Suppose $a_1, a_2, \ldots, a_n \in D$ are idempotent elements, i.e., $a_i^2 = a_i$ for all $1 \leq i \leq n$, and they satisfy

$$a_1 + a_2 + \cdots + a_n = 0$$

Prove that $a_1 = a_2 = \cdots = a_n = 0$.

6. **Escaping the matrix? (The determinant can't)** (3 points) Let $a_1, a_2, \ldots, a_n \in \mathbb{R}^n$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}^n$ be unit vectors.

For each i, define

$$c_i = a_i + \varepsilon_i b_i,$$

where $\varepsilon_i \in \{+1, -1\}$. Let C be the $n \times n$ matrix whose columns are $c_1, c_2, \ldots, c_n$.

Show that there exists a choice of signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{\pm 1\}$ such that

$$|\det(C)| \leq 2^{n/2}$$

7. **This one looks odd** (3 points) Prove that every odd polynomial function of degree $\leq 2n-1$ can be written as

$$P(x) = a_1 \binom{x}{1} + a_2 \binom{x+1}{3} + \cdots + a_n \binom{x+n-1}{2n-1}$$

where $\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}$

8. **Invert it** (3 points) For each positive integer n, let $M(n)$ be the $n \times n$ matrix whose (i, j) entry is equal to 1 if $i + j$ is divisible by j, and equal to 0 otherwise. Prove that $M(n)$ is invertible if and only if $n + 1$ is square-free.
(An integer is square-free if it is not divisible by the square of an integer larger than 1.)

Learning Resource for Problems 4 and 5:
Refer Putnam and Beyond by Razvan Gelca and Titu Andreescu - Topics 2.4.2 and 2.4.3