Mathematics Guild
Problem Set - 3

Challenge posed on: 08/07/2025 **Challenge conquered by: 14/07/2025**

1 Overview

- **Topics focused:**
  - Algebra
  - Polynomials
  - Functional Equations
- **Challengers:**
  - Raghav Iyengar
  - V Sriram
  - Arun Radhakrishnan
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Happy solving :)

2 Problems

1. **Functional Inequality on Natural Numbers** Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, the following inequality holds:

$$2n + 2024 \leq f(f(n)) + f(n) \leq 2n + 2026.$$

2. **Too cool to be in GP** Let $P(x)$ be a cubic polynomial with integer coefficients, and suppose that all three roots of $P(x)$ are positive irrational numbers. Prove that the roots cannot form a geometric progression.

3. **This is more confusing than Pokemon XYZ** The Society against Probability is a violent anarchist group of criminals in the country of Mathematica. Not liking leaving anything to chance, they are dead set on taking over the country by any means necessary. You are tasked by the government to infiltrate the society and bring them down.
   You've learnt that Agent Z, the current Axiom-master General of the group, has set up a new security system to prevent spies from entering the headquarters. He maintains a secret function $f : R \to R$ that is used to generate the passkey. On the nth day of the month, the passkey to enter the building is given by f(n).
   Your intel tells you that the function satisfies a special condition:

$$x(f(x+y) - f(x-y)) - y(f(x+y) + f(x-y)) = 8xy(x^4 - y^4)$$

   You also know that on the first day of this month, someone successfully used the passkey 2025. Can you find the secret function f, and help save the country?

4. **Even more confusing than Pokemon XYZ** Before you could finish solving the previous problem, Agent Z found about the spies you planted in his company. He's revamped the security system and is choosing a completely different type of function now. To save the country, you now must determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy + f(x)) + f(y) = xf(y) + f(x + y)$$

   for all real numbers x and y.

5. **Be real.** Suppose $a_0, a_1, \ldots, a_{100}$ are positive real numbers. For each $k \in \{0, 1, 2, \ldots, 100\}$, consider the polynomial

$$P_k(x) = a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k$$

where the indices are taken modulo 101, i.e., for any integer $i \in \{1, 2, \ldots, 100\}$, we define

$$a_{100+i} = a_{i-1}.$$

Prove: it is impossible for all 101 polynomials $P_0(x), P_1(x), \ldots, P_{100}(x)$ to have all their roots real.

6. **Add and Subtract** Find all polynomials f(x) and g(x), such that

$$\frac{d}{dx}(|f(x)| + g(x)) = 3^{\left(1+\frac{x-3}{|x-3|}\right)} \ \forall\ x \neq 3$$

Given: $g(0) = 12$