Mathematics Guild
Solutions to Problem Set - 2

Challenge posed on: 01/07/2025 **Challenge conquered by: 07/07/2025**

1 Overview

- **Topics focused:**
  - Equations
  - Geometry
  - Number Theory
  - Game Theory
- **Challengers:**
  - Pranjal Varshney
  - Pratyaksh Jain
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Happy solving :)

2 Problems

1. **Functional Equations** (a) Set x=y=0: $f(0) = f^2(0)$, i.e, f(0) = 0 or 1. Fow (0,y): $f(y^2) = y^2 + f(0)$ and for (x,0): $f(x^2) = f^2(x) - 2xf(0)$.

   Combine the two cases: $f^2(x) - 2xf(0) = x^2 + f^2(0)$, i.e $f^2(x) = (x + f(0))^2$.

   Method 1: Substitute in the original equation:

   $$f(y) = \frac{f^2(x) - f((x-y)^2) + y^2}{2x} = \frac{(x + f(0))^2 - (x-y)^2 - f^2(0) + y^2}{2x} = y + f(0)$$

   Method 2: For (x,x): $f(0) = f^2(x) - 2xf(x) + x^2 = (f(x) - x)^2$, i.e $f(x) = x \pm f(0)$. We already know that $f^2(x) = (x + f(0))^2$, i.e $f(x) = \pm(x + f(0))$. From the two we can conclude that f(x) = x + f(0).

   i.e **f(x) = x or x + 1**.

   (b) *For (0,b)*: $f(0)+2f(b) = f(f(b))$. Thus the original equation can be rewritten as $f(2a)+2f(b) = f(0) + 2f(a + b)$.

   Now for (1,b): $(f(b + 1) - f(b)) = \frac{f(2)-f(0)}{2}$ = const, i.e f(x) is linear. Take $f(b) = kb + f(0)$,

   Take (0,b) again: $f(0) + 2kb + 2f(0) = k^2b + kf(0) + f(0)$. Equate b coefficients: k = 0 or 2. Equate constant terms: $(k - 2)f(0) = 0$.
   Thus **f(x) = 0 or 2x+c**

2. **The one that's too big to read** Observe that $p_1 = 1$... let us also set $p_0 = 1$. For any n+1 (with positive n), we suppose that the first person occupies seat 'k'.
If $K = 1$: There is only one cup holder available to the first person and after that the same process starts with n people and n seats.
If $K > 1$: we only consider the case in whic the person chooses the cup holder to their right (the other one is impossible)... we repeat the process with the groups of **k-1** seats at the left and **n-k+1** seats at the right.

   Thus

$$p_{n+1} = \frac{1}{n+1} \times p_n + \frac{1}{n+1} \times \frac{1}{2} \times \sum_{k=2}^{n+1} p_{k-1} \times p_{n-k+1}$$

   On further simplification:

$$p_{n+1} = \frac{1}{2(n+1)} \times p_n + \frac{1}{2(n+1)} \times \sum_{k=0}^{n} p_k p_{n-k}$$

   uptil here counts as partial solve

   Let us consider the power series

$$P(x) = \sum_{0}^{\infty} p_n x^n$$

   Since it satisfies the recurrence relation

$$\frac{dP}{dx} = \sum_{n=0}^{\infty}(n+1)p_{n+1}x^n = \sum_{n=0}^{\infty} \frac{1}{2} p_n x^n + \sum_{n=0}^{\infty} \frac{1}{2}\left(\sum_{k=0}^{n} p_k p_{n-k} x^n\right)$$

   i.e

$$\frac{dP}{dx} = \frac{P(x)}{2} + \frac{P^2(x)}{2}$$

   Solving this differnetial equation we get $P(x) = \frac{e^{x/2}}{2-e^{x/2}}$

   Thus $p_1 + p_2 + \cdots = P(1) - 1 = \frac{2\sqrt{e}-2}{2-\sqrt{e}}$

3. **One in a million** **Greedy approach**: Suppose we have picked T = $\{t_1, t_2, \ldots, t_n\}$ where n is as large as possible (i.e it is impossible to add any more elements to T. That means for each $t \in \{1, 2, \ldots, 10^6\}$, either $t \in T$ or there exists two distinct elements a,b in A and $t_i \in T$ such that $t + a = t_i + b$ or

$$t = t_i + b - a$$

   There exist —T—*—A—*(—A—-1) = n*101*100 values of RHS in the above equation, and n values in the set T. Thus the sum:

$$101 * 100 * n + n \geq 10^6$$

   which implies $n > 99$

4. **A small gift to Sriram** Let us assume two vertices A and B follow this rule. Then the sum $\angle CAB + \angle DAB > 180^0$ and $\angle CBA + \angle DBA > 180^0$, i.e in triangles ABC and ABD, we have 4 angles who's sum is greater than 360^0.

5. **10 years after independence** (3 points) Pranjal was a freedom fighter and loved Math. In the year 1957, he came up with a challenge and posed it to his fellow freedom fighters. Unfortunately none of them could solve it. 68 years later(present day), he gives this challenge to you. How many minimum numbers raised to the fourth power are required, such that their sum is 1957^{1957}. The numbers are not necessarily distinct.

Solution: We can 1st show that 1957^{1957} cannot be a sum of 4 or lesser perfect fourths.

Proof :
Let $a_1^4 + a_2^4 + a_3^4 + a_4^4 = 1957^{1957}$. We know for any number a, $a^4 \equiv 0, 1 \mod 16$. Taking mod 16 on RHS,

$$1957^{1957} \equiv 5^{1957} \equiv 625^{489} \times 5 \equiv 5 \mod 16$$

LHS $\equiv 0, 1, 2, 3, 4 \mod 16$. Hence 1957^{1957} cannot be the sum of 4 perfect fourths or lesser.

Lets, prove it is the sum of 5 perfect fourths.

$$\begin{aligned}
1957^{1957} &= 1957 \times 1957^{1956} = 1957 \times (1957^{489})^4 \\
&= (625 + 625 + 625 + 81 + 1) \times (1957^{489})^4 \\
&= (5^4 + 5^4 + 5^4 + 3^4 + 1^4) \times (1957^{489})^4 \\
&= (5 \times 1957^{489})^4 + (5 \times 1957^{489})^4 + (5 \times 1957^{489})^4 + (3 \times 1957^{489})^4 + (1 \times 1957^{489})^4
\end{aligned}$$

Hence the answer is 5.

6. **A small gift to Sriram part 2** (3 points) One fine summer afternoon, Sriram was sitting on his table proving Geometry lemmas when suddenly his doorbell rang. On opening the door Sriram found a letter, which contained a geometry problem, on his doorstep. This is what was contained in the letter : A Right triangle PQR has its right angle at R and $\angle QPR = \theta$. You need to choose $S \in PQ$ such that $|PR| = |PS| = 1$ and $T \in QR$ such that $\angle RST = \theta$.

Now, the perpendicular drawn to QR at T meets PQ at U. Find

$$\lim_{\theta \to 0} TU.$$

Solution: 1st do angle chasing, then use sine rule :

$$\frac{|CD|}{1} = \frac{\sin\theta}{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)} \Rightarrow |CD| = \frac{\sin\theta}{\cos\left(\frac{\theta}{2}\right)}$$

$$\frac{|CE|}{|CD|} = \frac{\sin\theta}{\sin\left(\frac{\pi - 3\theta}{2}\right)} \Rightarrow |CE| = \frac{\sin^2\theta}{\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$\frac{|ED|}{|CD|} = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{3\theta}{2}\right)} \Rightarrow |ED| = \frac{\sin\left(\frac{\theta}{2}\right)\sin\theta}{\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$|BE| = \frac{\sin\left(\frac{\theta}{2}\right)\sin\theta}{\sin\left(\frac{3\theta}{2}\right)\cos\theta}, \qquad |EC| = \frac{\sin^2\theta}{\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \triangle ABC \sim \triangle FBE$$

$$\frac{|EF|}{1} = \frac{|BE|}{|BC|} \Rightarrow \frac{1}{|EF|} = \frac{|BE| + |EC|}{|BE|}$$

$$\Rightarrow \frac{1}{|EF|} = 1 + \frac{|EC|}{|BE|}$$

$$\Rightarrow \frac{1}{|EF|} = 1 + \frac{\sin\theta\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)\cos\theta}$$

$$\Rightarrow \frac{1}{|EF|} = 1 + \frac{\sin\theta\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)\cos\theta} \underset{\theta\to 0}{\Rightarrow\!\!\longrightarrow} 1 + 2 = 3$$

$$\boxed{\lim_{\theta\to 0} |EF| = \frac{1}{3}}$$

7. **Warm-Up and Workout** (a) (2 points) Solve the system

$$\begin{aligned}(x+y)(x+y+z) &= 18,\\ (y+z)(x+y+z) &= 24,\\ (z+x)(x+y+z) &= 30.\end{aligned}$$

(b) (3 points) For what whole numbers w is $w^4 + 4^w$ composite?

Solution:

(a) i wonder what the solution to this is. Is the problem even solved :').

(b) Let $f(n) = n^4 + 4^n$, with $n > 1$. Then $f(n) > 2$ is *even* when n is *even*, and hence *composite* in these cases. Henceforth, assume n is *odd*. Now

$$\begin{aligned}n^4 + 4^n &= \left(n^2 + 2^n\right)^2 - \left(2^{(n+1)/2}\cdot n\right)^2\\ &= \left(n^2 + 2^{(n+1)/2}n + 2^n\right)\left(n^2 - 2^{(n+1)/2}n + 2^n\right),\end{aligned}$$

and the *smaller* factor

$$n^2 - 2^{(n+1)/2} + 2^n = \left(n - 2^{(n-1)/2}\right)^2 + 2^{n-1} \geq 2^{n-1} > 1.$$

Hence $f(n)$ is *composite* even when n is *odd*. ■

8. **An adaptation** (3 points) Pranjal likes to play this game with Ronak, where Ronak thinks of a 4 digit number in his mind, and Pranjal is given 10 tries to guess the number. Pranjal gets an idea and decides to challenge Ronak with another game. He thinks of a number in the set $A = \{1, 2, 3, 4, 5, 6\}$ and Ronak has to guess what it is. If he is able to guess it correctly, he wins but if he guesses it incorrectly, Pranjal will always increase or decrease his number by 1 (keeping it in the set A) before Ronak's next guess.

Pranjal gives him a limit for his tries. What is the smallest limit k that Pranjal can impose such that Ronak is guaranteed to win within k guesses? With what strategy should Ronak play?

Solution: The answer is $k = 8$. A sequence of 8 guesses which is guaranteed to win is:

$$2, 3, 4, 5, 5, 4, 3, 2.$$

The reason for this is as follows. Suppose that Pranjal's initial number is even.

- If the first guess 2 is incorrect, Pranjal's number at that time is either 4 or 6, and must change to either 3 or 5.

- Then if the second guess 3 is incorrect, Pranjal's number at that time is 5, and must change to either 4 or 6.
- Then if the third guess 4 is incorrect, Pranjal's number at that time is 6, and must change to 5.
- So the fourth guess 5 is guaranteed to be correct.

Therefore, if Ronak has not won within the first four guesses, it must be that Pranjal's initial number was odd. Since her number changes parity after each unsuccessful guess, her number will again be odd after four guesses.

- Then if the fifth guess 5 is incorrect, Pranjal's number at that time is either 1 or 3, and must change to either 2 or 4.
- Then if the sixth guess 4 is incorrect, Pranjal's number at that time is 2, and must change to either 1 or 3.
- Then if the seventh guess 3 is incorrect, Pranjal's number at that time is 1, and must change to 2.
- So then the eighth guess 2 is guaranteed to be correct.

Now we must show that no sequence of fewer than 8 guesses is guaranteed to win. It is enough to show that no sequence where one of the *internal* numbers 2, 3, 4, 5 occurs fewer than twice is guaranteed to win. Suppose that the number 2 occurs fewer than twice in the sequence of guesses — the argument for other internal numbers is analogous. Pranjal's number could conceivably alternate between 2 and one of the neighbours of 2 — that is, 1 or 3, but not necessarily the same neighbour each time. In this case, even if there is one occasion when Ronak guesses 2, Pranjal's starting parity could have been such that she is at one of the neighbours 2 at that time; and on any occasion when Ronak guesses one of the neighbours of 2, Pranjal could be either at 2 or at the other neighbour of 2. So it is possible that Pranjal evades all of Ronak's guesses.

Comment. One can generalise to show that if we instead take $A = \{1, 2, 3, \ldots, n\}$ for $n \geq 3$, then the smallest number of guesses such that Ronak can guarantee to win is $2n - 4$.

Say hello to Tralalero Tralala (1 point) (just kidding lmao)

Solution: Hello Tralalero Tralala :)