Mathematics Guild
Problem Set - 2

Challenge posed on: 01/07/2025 **Challenge conquered by: 07/07/2025**

1 Overview

- **Topics focused:**
  - Equations
  - Geometry
  - Number Theory
  - Game Theory
- **Challengers:**
  - Pranjal Varshney
  - Pratyaksh Jain
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Happy solving :)

2 Problems

1. **Functional Equations** Find all possible functions:
   (a) (2 points) $f : R \to R$ such that $f((x-y)^2) = f(x)^2 - 2xf(y) + y^2$
   (b) (2 points) $f : Z \to Z$ (where Z is the set of integers) such that $\forall$ integers (a,b): $f(2a) + 2f(b) = f(f(a+b))$

2. **The one that's too big to read** (3 points) A theater in Phoenix mall has n seats in a row, numbered left to right from 1 to n. There is a cup holder between any two seats and on the right of seat n (i.e, seat 1 has one cup holder and all other seats have two cup holders adjacent to them). There are n people in line, each holding a drink. They enter one by one, choose an available seat uniformly at random and do the following:
   - If they have two empty cup holders available, they choose one uniformly at random.
   - If they have only one empty cup holder available, they use it.
   - If they have no empty cup holders they hold the drink in their hands.

   Let p_n be the probability that all n people place their drink in a cup holder.
   Determine $p_1 + p_2 + p_3 + \ldots$.

3. **One in a million** (3 points) Let A be a 101-element subset of the set $S = 1, 2, \ldots, 10^6$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \quad | \quad x \in A\}, \quad j = 1, 2, \ldots, 100$$

   are pairwise disjoint.

4. **A small gift to Sriram** (3 points) Sriram is obsessed with Tetrahedrons and begins his design with a vertex A such that the sum of any two plane angles (i.e all pairs of $\angle BAC, \angle BAD$ and $\angle CAD$) is greater than 180 degrees. However he is not able to design a tetrahedron with more than one such vertex. Prove that what Sriram is trying to achieve isn't possible.

5. **10 years after independence** (3 points) Pranjal was a freedom fighter and loved Math. In the year 1957, he came up with a challenge and posed it to his fellow freedom fighters. Unfortunately none of them could solve it. 68 years later(present day), he gives this challenge to you. How many minimum numbers raised to the fourth power are required, such that their sum is 1957^{1957}. The numbers are not necessarily distinct.

6. **A small gift to Sriram part 2** (3 points) One fine summer afternoon, Sriram was sitting on his table proving Geometry lemmas when suddenly his doorbell rang. On opening the door Sriram found a letter, which contained a geometry problem, on his doorstep. This is what was contained in the letter : A Right triangle PQR has its right angle at R and $\angle QPR = \theta$. You need to choose $S \in PQ$ such that $|PR| = |PS| = 1$ and $T \in QR$ such that $\angle RST = \theta$.

   Now, the perpendicular drawn to QR at T meets PQ at U. Find

   $$\lim_{\theta \to 0} TU.$$

7. **Warm-Up and Workout** (a) (2 points) Solve the system

   $$\begin{aligned}(x+y)(x+y+z) &= 18,\\ (y+z)(x+y+z) &= 24,\\ (z+x)(x+y+z) &= 30.\end{aligned}$$

   (b) (3 points) For what whole numbers w is $w^4 + 4^w$ composite?

8. **An adaptation** (3 points) Pranjal likes to play this game with Ronak, where Ronak thinks of a 4 digit number in his mind, and Pranjal is given 10 tries to guess the number. Pranjal gets an idea and decides to challenge Ronak with another game. He thinks of a number in the set $A = \{1, 2, 3, 4, 5, 6\}$ and Ronak has to guess what it is. If he is able to guess it correctly, he wins but if he guesses it incorrectly, Pranjal will always increase or decrease his number by 1 (keeping it in the set A) before Ronak's next guess.

   Pranjal gives him a limit for his tries. What is the smallest limit k that Pranjal can impose such that Ronak is guaranteed to win within k guesses? With what strategy should Ronak play?

Say hello to Tralalero Tralala (1 point) (just kidding lmao)