Mathematics Club
Solutions to Problem Set - 1

Challenge posed on: 24/06/2025 **Challenge conquered by: 29/06/2025**

1 Overview

- **Topics focused:**
  - Inequalities
  - Number Theory
  - Game Theory
- **Challengers:**
  - Pranjal
  - Pratyaksh
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard

2 Problems

1. **An inequality** Let

$$x = \frac{a}{\sqrt[3]{abc}}, \quad y = \frac{b}{\sqrt[3]{abc}}, \quad z = \frac{c}{\sqrt[3]{abc}}.$$

We need to show that

$$\left(1 + \frac{x}{y}\right)\left(1 + \frac{y}{z}\right)\left(1 + \frac{z}{x}\right) \geq 2\left(1 + x + y + z\right),$$

or, since $xyz = 1$,

$$(x+y)(y+z)(z+x) \geq 2 + 2(x+y+z). \tag{1}$$

which is equivalent to

$$(x+y+z)(xy+yz+zx-2) - xyz \geq 2. \tag{2}$$

Since $xyz = 1$, the AM-GM inequality implies

$$x+y+z \geq 3 \quad \text{and} \quad xy+yz+zx \geq 3.$$

So (2) follows.

2. **A unique competition** Let the total number of students be n+2 (n unkown people). Let the n unknown people score k each.

Now, notice in every match exactly 1 point is distributed. Total number of matches : $\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$.

So total score : $nk + 8 = \frac{(n+2)(n+1)}{2}$. You get $n^2 - (2k-3)n - 14 = 0$. And n is an integer. So $n(n-2k+3) = 14$. n is a factor of 14. Checking for n=1, 2, 7 and 14. n=1 and 2 gives k negative. n=7 gives k=4 and n=14 gives k=8 which are possible. So n can be 7 and 14. Answer : 98

3. **Inequality again?!** The inequality is equivalent to

$$\frac{n^n}{n+1} < \binom{n}{k} k^k (n-k)^{n-k} < n^n,$$

which suggests investigating the binomial expansion of

$$n^n = ((n-k)+k)^n = \sum_{i=0}^{n} \binom{n}{i} (n-k)^{n-i} k^i.$$

The $(k+1)$th term T_{k+1} of the expansion is $\binom{n}{k} k^k (n-k)^{n-k}$, and all terms in the expansion are positive, which implies the right inequality.

Now, for $1 \leq i \leq n$,

$$\frac{T_{i+1}}{T_i} = \frac{\binom{n}{i}(n-k)^{n-i}k^i}{\binom{n}{i-1}(n-k)^{n-i+1}k^{i-1}} = \frac{(n-i+1)k}{i(n-k)},$$

and

$$\frac{T_{i+1}}{T_i} > 1 \iff (n-i+1)k > i(n-k) \iff i < k + \frac{k}{n} \iff i \leq k.$$

This means that

$$T_1 < T_2 < \cdots < T_{k+1} > T_{k+2} > \cdots > T_{n+1},$$

that is, $T_{k+1} = \binom{n}{k} k^k (n-k)^{n-k}$ is the largest term in the expansion. The maximum term is greater than the average, which is the sum n^n divided by the quantity $n+1$, therefore

$$\binom{n}{k} k^k (n-k)^{n-k} > \frac{n^n}{n+1},$$

as required.

4. **Take a break** Since $a_{2025-i} = \frac{2025-i}{2025} = 1 - \frac{i}{2025} = 1 - a_i$ and

$$1 - 3a_i + 3a_i^2 = (1 - 3a_i + 3a_i^2 - a_i^3) + a_i^3 = (1-a_i)^3 + a_i^3 = a_{101-i}^3 + a_i^3,$$

we have, by replacing i by $2025 - i$ in the second sum,

$$2S = S + S = \sum_{i=0}^{2025} \frac{a_i^3}{a_{2025-i}^3 + a_i^3} + \sum_{i=0}^{2025} \frac{a_{2025-i}^3}{a_i^3 + a_{2025-i}^3} = \sum_{i=0}^{2025} \frac{a_i^3 + a_{2025-i}^3}{a_{2025-i}^3 + a_i^3} = 2026,$$

so $S = 1013$.

5. **Modulo?** (a) Trivial solution (0,0,0,0)

Apply mod 6 on both sides, we notice that n is a multiple of 6... let $n = 6n_1$ to reduce the equation to $6a^2 + 3b^2 + c^2 = 30{n_1}^2$. Further apply mod 3 and let $c = 3c_1$ to get $2a^2 + b^2 + 3{c_1}^2 = 10{n_1}^2$.

Thus b,c and c_1 have the same parity. Further $b^2 + 3{c_1}^2 \equiv 2({n_1}^2 - a^2) \equiv 0 \bmod 4$, (since even and odd numbers are 0 and 1 mod 4 respectively) i.e a and n_1 have the same parity.

a and n_1 having the same parity guarantees that $2(5{n_1}^2 - a^2) \equiv 0 \bmod 8$. Thus b and c_1 are both even. Further $b^2 + 3{c_1}^2 \equiv 0 \bmod 16$, hence a and n_1 are also even

We can proceed indefinitely... hence there are no other solutions.

(b) RHS $\equiv$ 1,3 or 9 modulo 13, and LHS $\equiv$ 11,12,2,7,1,10 or 8 modulo 13. Thus both need to be equal to 1 mod 13, i.e y must be a multiple of 3. Let y = 3k for a positive integer k. Now for $z = 3^k$ we have $x^2 + 11 = z^3$

Factorize the LHS and express RHS in terms of these factors, we get $x \pm \sqrt{11} = (\frac{a \pm \sqrt{11}}{2})^3$ (+ on LHS corresponds to + on RHS) where a and b are either both even or both odd. Equate the inaginary parts of any one term, we get $\pm 2^3 = 3a^2b - 11b^3$, hence b divides 2^3. On checking individual cases, we get $a = \pm 1$, $b = \pm 1$, i.e x = 4, y = 3 is the only solution.

(c) By observation we can find that x=1 is a solution. For $x > 1, x!$ is even, hence $3^{x!} \equiv 1$ mod 4. This implies that last digit of $2^{3^{x!}}$ is 2 $\rightarrow$ last digit of LHS is 3. But last digit of an even power or 3 is either 1 or 9.

6. **Subparts once again** (a) Substitute (b*a,b) in place of (a,b). Now we get ((b*a)*b)*(b*a) = b **or** a*(b*a) = b.

(b) Since S is non-empty, for some $i = j \in S$, we get $\frac{2i}{gcd(i,i)} = \frac{2i}{i} = 2 \in S$. Now consider the following:

Odd numbers: If odd $k \in S$, take i=k, j=2: $\frac{k+2}{gcd(k,2)} = k + 2 \in S$. Further if $k + 2 \in S$, $k + 4 \in S$ and so on. Thus if an odd number is in S then S has infinite elements.

Even numbers other than 2: If even $k \neq 2 \in S$, take i=k, j=2, $\frac{k+2}{2} \in$. Since this needs to be an even number, no multiples of 4 are allowed in S. Thus the denominator must always be 2, i.e, for any two numbers $\in S$, their average $\in S$. Eventually we will encounted a multiple of 4 or an odd number as average. Thus it can be concluded that **S can have only one element**.

7. **One more game** $O\triangle\square$ In both cases existence of a real root can be ensured.

- **Pranjal has the winning strategy**: Pratyaksh can pick 6 different quadratics resulting in 3 different determinants. But Pranjal can ensure all of the determinants are > 0. Let us assume Pranjal picks $a > 0$:

  If Pratyaksh picks $b > 0$: Pranjal can choose any $c < -2\sqrt{ab}$ (we need $c^2 - 4ab > 0$, the other two are > 0 for any $c < 0$).
  If Pratyaksh picks $b < 0$: Pranjal can choose any $0 < c < \frac{b^2}{4a}$ (such that $b^2 - 4ac > 0$, the other two are > 0 for any $c > 0$).

- **Pratyaksh has the winning strategy**: Pratyaksh simply picks b = -a and the polynomial $ax^2 + cx + b$. Dividing by a we get $x^2 + kx - 1$, determinant $k^2 + 4 > 0$.