Mathematics Club
Contingent Problem Set - 1

Challenge posed on: 24/06/2025 **Challenge conquered by: 29/06/2025**

1 Overview

- **Topics focused:**
  - Inequalities
  - Number Theory
  - Game Theory
- **Challengers:**
  - Pranjal
  - Pratyaksh
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Problems are not as scary as they look. You just need to find the correct key to open the lock.
- Happy solving :)

2 Problems

1. **An inequality** Give that x, y, z are positive real numbers. You must prove that

$$\left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)\left(1+\frac{z}{x}\right) \geq 2\left(1+\frac{x+y+z}{\sqrt[3]{xyz}}\right)$$

2. **A unique competition** Pranjal and Pratyaksh participated in a math competition that contained 1 v 1 battles. Several other people participated in it who they didn't know. Each participant battles against each of the other participants. In each battle, the winner was awarded 1 point, the loser got 0 points, and each players earned $\frac{1}{2}$ a point in case of a tie. Pranjal and Pratyaksh managed to get 8 points in total, and it was found that all other particants got the same number of points. What is the product of the possible number of participants in the competition?

3. **Inequality again?!** Let n, k be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}}.$$

4. **Take a break** Find $S = \sum_{i=0}^{2025} \frac{a_i^3}{1-3a_i+3a_i^2}$ for $a_i = \frac{i}{2025}$.

5. **Modulo?** (a) **Deja vu**: Find all possible integers (a,b,c,n) that satisfy $6(6a^2+3b^2+c^2) = 5n^2$.
   (b) **Die-phantine**: Find all positive integers (x,y) that satisfy the equation $x^2+11 = 3^y$
   (c) **Why is the last one easier?** Find all positive integers x such that $3^{2^{x!}} = 2^{3^{x!}} + 1$

6. **Subparts once again** (a) Given a set S and a Binary Operation * on S. Assume that (a*b)*a = b $\forall a, b \in S$. Prove that a*(b*a) = b $\forall a, b \in S$

(b) Determine all finite nonempty sets S of positive integers such that

$$\frac{i+j}{gcd(i,j)}$$

is an element of $S \forall i, j \in S$

7. **One more game** $O\triangle\square$ Pranjal and Pratyksh play a game. First Pranjal chooses a non-zero real number 'a'. Then Pratyaksh chooses a non-zero real number 'b'. Pranjal again chooses a non-zero real number 'c' and then Pratyaksh chooses a quadratic polynomial with the coefficients a,b,c in some order:

- Suppose Pranjal wins if the polynomial has a real root and Pratyaksh wins otherwise.
- Suppose Pratyaksh wins if the polynomial has a real root and Pranjal wins otherwise.

Find out which player wins in both cases.

8. **Guess who's back** An icosahedral (20 faces) die is rolled twice and the numbers A and B are obtained. *Bombardino Crocodilo* and *tung tung tung Sahur* once again take turns drawing from a pile of 21 coins, but this they can only draw in steps of 1, A or B. If both players play optimally, what is the probability that Sahur wins, given that Sahur always draws first.