i Opener Problem Set

THE MATHEMATICS CLUB

Instructions:

- You are given this paper at the start of the session and you are supposed to answer these based on what you'll learn throughout the session.
- You will be given 30 minutes after the session to solve the questions in this paper.
- You are expected to solve the problems subjectively: the method matters more than the final answer.

1. Using complex numbers find out the simplified result of the summation

$$1 + \cos(\theta) + \cos(2\theta) + ... + \cos(n\theta)$$

Solution:

Let $S = 1 + z + z^2 + \cdots + z^n$. Then

$$(1 - z)S = S - zS = (1 + z + z^2 + \cdots + z^n) - (z + z^2 + z^3 + \cdots + z^{n+1}) = 1 - z^{n+1}$$

Substituting $z = e^{i\theta}$ gives

$$1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}}.$$

Multiplying the numerator and denominator by $ie^{-i\theta/2}$ yields

$$1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{ie^{-i\theta/2} - ie^{(2n+1)i\theta/2}}{ie^{-i\theta/2} - ie^{i\theta/2}} = \frac{ie^{-i\theta/2} - ie^{(2n+1)i\theta/2}}{2\sin(\theta/2)}.$$

Then

$$\begin{aligned} 1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta &= \mathrm{Re}(1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta}) \\ &= \mathrm{Re}\left(\frac{ie^{-i\theta/2} - ie^{(2n+1)i\theta/2}}{2\sin(\theta/2)}\right) \\ &= \frac{\sin(\theta/2) + \sin[(2n+1)\theta/2]}{2\sin(\theta/2)} = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}. \end{aligned}$$

2. For which values of $a, b \in \mathbb{R}$, $f(z) = ax^2 + by^2 - 2xyi$ is holomorphic, where $z = x + iy$?

Solution:

Given:

$$f(z) = ax^2 + by^2 - 2xyi = u(x, y) + iv(x, y),$$

so

$$u(x, y) = ax^2 + by^2, \quad v(x, y) = -2xy.$$

Cauchy-Riemann equations:

$$u_x = v_y, \quad u_y = -v_x.$$

Compute derivatives:

$$u_x = 2ax, \quad u_y = 2by, \quad v_x = -2y, \quad v_y = -2x.$$

Apply CR equations:

$$u_x = v_y \implies 2ax = -2x \implies a = -1 \quad (\text{for all } x)$$

$$u_y = -v_x \implies 2by = 2y \implies b = 1 \quad (\text{for all } y)$$

Conclusion:

$$\boxed{a = -1,\ b = 1}.$$

The holomorphic function is then

$$f(z) = -x^2 + y^2 - 2xyi.$$

3. Calculate: $\oint_C \frac{e^z}{z^4} dz$, C is a unit circle centered at the origin.

Solution:

To evaluate

$$\oint_C \frac{e^z}{z^4}\, dz,$$

where C is the unit circle $|z| = 1$.

Using Cauchy's integral formula for derivatives:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz,$$

where f is analytic inside and on C, and a is inside C.

Here, $f(z) = e^z$, $a = 0$, and $n + 1 = 4 \implies n = 3$.

$$\oint_C \frac{e^z}{z^4}\, dz = 2\pi i \frac{f^{(3)}(0)}{3!}.$$

Computing the derivative

$$f^{(3)}(z) = \frac{d^3}{dz^3} e^z = e^z \implies f^{(3)}(0) = 1.$$

$$\oint_C \frac{e^z}{z^4}\,dz = 2\pi i \frac{1}{3!} = \frac{2\pi i}{6} = \frac{\pi i}{3}.$$

$$\boxed{\frac{\pi i}{3}}$$

4. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Also assume $|f(z)| \leq 6\left|z^2\right|$.

   Given, $f(1) = 2 + 3i$, find $f(3 + 2i)$.

Solution:

The function

$$g(z) = \frac{f(z)}{z^2}, \quad \left(g(0) := \lim_{z\to 0} \tfrac{f(z)}{z^2}\right)$$

is entire and bounded. By Liouville's theorem, g is constant, say $g(z) = a$ with $|a| \leq 6$. Hence

$$f(z) = az^2.$$

Since $f(1) = 2 + 3i$, we get $a = 2 + 3i$. Therefore

$$f(z) = (2 + 3i)z^2.$$

Finally,

$$f(3 + 2i) = (2 + 3i)(3 + 2i)^2 = (2 + 3i)(5 + 12i) = -26 + 39i.$$

Answer: $f(3 + 2i) = -26 + 39i.$

5. Calculate $\oint_C \frac{(z-4)(z^2+4)-3}{z(z^2+4)}dz$ where C is a circle of radius 3 centered at $(0, 0)$.

Solution:

The poles of the integrand are at $z = 0,\ 2i,\ -2i$, all inside $|z| = 3$. By the residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, \mathrm{pole}).$$

$$\mathrm{Res}(f, 0) = \lim_{z\to 0} z \cdot \frac{(z-4)(z^2+4)-3}{z(z^2+4)} = \frac{(0-4)(0^2+4)-3}{4} = -\tfrac{19}{4}.$$

For $z = 2i$:

$$N(z) = (z-4)(z^2+4) - 3, \quad D(z) = z(z^2+4), \quad D'(z) = 3z^2 + 4.$$

$$N(2i) = -3, \quad D'(2i) = -8 \quad \Rightarrow \quad \mathrm{Res}(f, 2i) = \frac{-3}{-8} = \tfrac{3}{8}.$$

Similarly, at $z = -2i$:

$$\text{Res}(f, -2i) = \tfrac{3}{8}.$$

Thus,

$$\sum \text{Res} = -\tfrac{19}{4} + \tfrac{3}{8} + \tfrac{3}{8} = -\tfrac{19}{4} + \tfrac{3}{4} = -4.$$

Therefore,

$$\oint_C f(z)\,dz = 2\pi i \cdot (-4) = -8\pi i.$$

$$\boxed{-8\pi i}$$

6. Compute $\displaystyle\sum_{k=1}^{13} \csc^2\left(\frac{k\pi}{14}\right)$.

Solution:

Step 1: Use a function with simple poles. Consider

$$f(z) = \pi\cot(\pi z),$$

which is meromorphic (A meromorphic function is a complex function that is holomorphic (analytic) everywhere in its domain except for a discrete set of isolated points, called poles, where the function goes to infinity) on $\mathbb{C}$ with simple poles at integers $z \in \mathbb{Z}$, each with residue 1.

Its derivative is

$$f'(z) = -\pi^2 \csc^2(\pi z).$$

Step 2: Consider the n-th roots of unity. Let

$$\omega = e^{2\pi i/n}, \quad n \in \mathbb{N},$$

and the polynomial

$$P(z) = z^n - 1 = \prod_{k=0}^{n-1} (z - \omega^k).$$

Taking logarithmic derivative gives

$$\frac{P'(z)}{P(z)} = \sum_{k=0}^{n-1} \frac{1}{z - \omega^k}.$$

Step 3: Substitute $z = 1$.

$$\sum_{k=1}^{n-1} \frac{1}{1-\omega^k} = \lim_{z\to 1}\left(\frac{P'(1)}{P(1)} - \frac{1}{z-1}\right) = \lim_{z\to 1}\left(\sum_{k=0}^{n-1} \frac{1}{z-\omega^k} - \frac{1}{z-1}\right)$$

Since $P(1) = 0$, we consider the limit

$$\lim_{z\to 1}\left[\frac{P'(z)}{P(z)} - \frac{1}{z-1}\right] = \frac{P''(1)}{2P'(1)}$$

using L'Hospital's rule. This gives a finite sum involving $\sum_{k=1}^{n-1} \frac{1}{(1-\omega^k)^2}$.

Step 4: Relate to $\csc^2$. Using

$$\csc^2\frac{k\pi}{n} = \frac{1}{\sin^2\frac{k\pi}{n}} = \frac{4}{|1-e^{2\pi ik/n}|^2} = \frac{4}{|1-\omega^k|^2},$$

we get

$$\sum_{k=1}^{n-1}\csc^2\frac{k\pi}{n} = 4\sum_{k=1}^{n-1}\frac{1}{|1-\omega^k|^2}.$$

Step 5: Use algebraic identity. It can be shown (via partial fractions or the above limits) that

$$\sum_{k=1}^{n-1}\csc^2\frac{k\pi}{n} = \frac{n^2-1}{3}.$$

Step 6: Apply for $n = 14$:

$$\sum_{k=1}^{13}\csc^2\frac{k\pi}{14} = \frac{14^2-1}{3} = \frac{195}{3} = 65.$$

$$\boxed{65}$$

7. Hemanth is very interested in the curve generated by the equation $\left|\frac{z+1}{z-1}\right| = 4$. Find out what the curve is.

Hemanth decides that he will integrate the value of the function z^2 along the curve and starts from the point $\left(\frac{5}{3}, 0\right)$, but he gets tired halfway and only computes the integral until the halfway point. Find the magnitude of the integral that he obtained.

Solution:

We prove that the following curve is a circle

$$|z+1| = r|z-1|$$

$$r^2(|z|^2 + 1 - z - \bar{z}) = (|z|^2 + 1 + \bar{z} + z)$$

$$r^2 - 1 = (1-r^2)|z|^2 + (z+\bar{z})(1+r^2)$$

$$r^2 - 1 = (x^2+y^2)(1-r^2) + 2x(1+r^2)$$

Clearly this is the equation of a circle whose center lies on the x axis, substituting r = 4 and solving for the points on the x axis we obtain $\left(\frac{5}{3}, 0\right)$ and $\left(\frac{3}{5}, 0\right)$

The second part can be solved using the fact that the integral of a closed loop is 0, thus the semicircular path between the two points on the x axis can be substituted with the straight line path between the two, that integral is easy to compute

$$\int_{\frac{3}{5}}^{\frac{5}{3}} x^2 dx \approx 1.47$$

8. A common problem encountered in physics is that of the damped harmonic oscillator which requires solving the differential equation

$$\frac{d^2x(t)}{dt^2} + 2\gamma\frac{dx(t)}{dt} + \omega_0^2 x(t) = F(t)$$

The green's function helps to find the solution to this differential equation where the green's function is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\gamma\omega} d\omega$$

Simplify the integral assuming the damping term γ to be 0, take $\omega_0 t$ to be $\frac{\pi}{2}$ and ω_0 as $\frac{1}{2}$

Solution:

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2} d\omega$$

Factor the denominator:

$$\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega)$$

The poles are at $\omega = \pm\omega_0$.
Partial fraction decomposition gives:

$$\frac{1}{\omega_0^2 - \omega^2} = \frac{1}{2\omega_0}\left(\frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega}\right)$$

For $t > 0$, closing the contour in the upper half-plane and applying the residue theorem gives:

$$G(t) = \frac{\sin(\omega_0 t)}{\omega_0}, \quad t > 0$$

For $t < 0$, closing the contour in the lower half-plane gives:

$$G(t) = 0, \quad t < 0$$

Final result:

$$G(t) = \begin{cases} \frac{\sin(\omega_0 t)}{\omega_0}, & t > 0 \\ 0, & t < 0 \end{cases}$$

Are you having fun? Want more?

1. For a triangle with vertices being the complex numbers a, b, c what is the general form of the points lying inside or on the boundary of the triangle. Write it in some parametric form.

Solution:

The general point of a triangle can be written as $pa + qb + rc$ given that $p + q + r = 1$ and p,q,r lie between 0 and 1. To prove this we fix the point b and first find the general form of a point between b and c. That would be of the form $\alpha b + (1 - \alpha)c$ where α lies between 0 and 1. Now we take a convex combination of a point between b and c and the point a to obtain $(1 - \beta)a + (1 - \alpha)\beta b + \alpha\beta c$. This is the same as the general form presented earlier and any point inside the triangle must lie between a and some point between vertices b and c and thus this is the general form of a point inside the triangle.

2. How many non real roots are there for $f(z) = 4z^5 - z^2 + 9$?

Solution:

To find the number of non-real roots of

$$f(z) = 4z^5 - z^2 + 9.$$

Since f has real coefficients, non-real roots occur in conjugate pairs. Thus, if r is the number of real roots, then the number of non-real roots is $5 - r$.

For real x, define

$$g(x) = 4x^5 - x^2 + 9.$$

Then

$$g'(x) = 20x^4 - 2x = 2x(10x^3 - 1).$$

Critical points: $x = 0$ and $x = \sqrt[3]{\frac{1}{10}} > 0$.

- If $x < 0$: then $2x < 0$ and $10x^3 - 1 < 0$, so $g'(x) > 0$. Hence g is increasing on $(-\infty, 0)$.
- If $0 < x < \sqrt[3]{1/10}$: then $2x > 0$ and $10x^3 - 1 < 0$, so $g'(x) < 0$. Hence g is decreasing here.
- If $x > \sqrt[3]{1/10}$: then $g'(x) > 0$. Hence g is increasing here.

$$\lim_{x\to-\infty} g(x) = -\infty, \qquad g(0) = 9 > 0.$$

Since g is strictly increasing on $(-\infty, 0)$, there is exactly one real root in $(-\infty, 0)$.

For $x \geq 0$, note that

$$g\left(\sqrt[3]{\frac{1}{10}}\right) = 4\left(\frac{1}{10}\right)^{5/3} - \left(\frac{1}{10}\right)^{2/3} + 9 \approx 8.87 > 0.$$

Thus $g(x) > 0$ for all $x \geq 0$, so there are no positive real roots.

Hence $f(z)$ has exactly one real root (negative), and therefore

$$\text{number of non-real roots} = 5 - 1 = 4.$$

$\boxed{4}$

3. Let $f(z) = u(x,y) + iv(x,y)$ be an holomorphic function, where $z = x + iy$ and u, v are real-valued functions. Given

$$u(x,y) = x^2 + y^2,$$

Find the holomorphic conjugate $v(x,y)$. Also, determine the value of $f(1-i)$.

Solution:

Given: $u(x,y) = x^2 + y^2$.

Cauchy-Riemann equations: For $f = u + iv$ to be holomorphic,

$$u_x = v_y, \quad u_y = -v_x.$$

Compute partial derivatives of u:

$$u_x = 2x, \quad u_y = 2y.$$

Then the CR equations require:

$$v_y = 2x, \quad v_x = -2y.$$

Check for consistency:

Integrating $v_y = 2x$ with respect to y gives

$$v(x,y) = 2xy + g(x),$$

where $g(x)$ is a function of x alone.

Then

$$v_x = 2y + g'(x),$$

but CR requires $v_x = -2y$. Hence

$$2y + g'(x) = -2y \implies g'(x) = -4y,$$

which is a contradiction, since $g'(x)$ depends only on x, but the right-hand side depends on y.

Answer:

No holomorphic conjugate $v(x,y)$ exists. Therefore, a holomorphic function $f(z) = u + iv$ with $u(x,y) = x^2 + y^2$ does not exist, and $f(1-i)$ cannot be evaluated.

4. Is $f(z) = \frac{1}{z}, z \in \mathbb{C} \setminus \{0\}$ analytic?

Solution:

$$f(z) = \frac{1}{z}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

Holomorphy: For $z \neq 0$ the complex derivative exists:

$$f'(z) = \lim_{h \to 0} \frac{1/(z+h) - 1/z}{h} = \lim_{h \to 0} \frac{-h}{z(z+h)h} = -\frac{1}{z^2}.$$

Hence f is differentiable at every $z \neq 0$, so f is holomorphic on $\mathbb{C} \setminus \{0\}$.

Singularity at 0: The point $z = 0$ is a simple pole of f, so f is not analytic at 0.

$$\boxed{f(z) = \tfrac{1}{z} \text{ is analytic on } \mathbb{C} \setminus \{0\} \text{ and has a simple pole at } 0.}$$

5. Check if $f(z) = z^3 + 2z$ is holomorphic? If yes, also calculate $f'(3+4i)$.

Solution:

Checking holomorphy: The function

$$f(z) = z^3 + 2z$$

is a polynomial in z. Since all polynomials are entire functions, $f(z)$ is holomorphic on $\mathbb{C}$.

Computing the derivative:

$$f'(z) = \frac{d}{dz}(z^3 + 2z) = 3z^2 + 2.$$

Evaluating at $z = 3 + 4i$:

$$(3+4i)^2 = 9 + 24i + 16i^2 = -7 + 24i,$$

$$f'(3+4i) = 3(-7+24i) + 2 = -21 + 72i + 2 = -19 + 72i.$$

Answer: $f(z)$ is holomorphic and $f'(3+4i) = -19 + 72i$.

6. Find the value: $\displaystyle\prod_{k=1}^{20} \left(1 - e^{2\pi i k/21}\right)$

Solution:

Step 1: Introduce roots of unity

Let

$$\omega = e^{2\pi i/21},$$

a 21st root of unity. Then the product can be written as

$$\prod_{k=1}^{20} (1 - \omega^k).$$

Step 2: Factorization of $x^n - 1$

We know that

$$x^{21} - 1 = \prod_{k=0}^{20} (x - \omega^k).$$

Step 3: Factor out $(x - 1)$ carefully

We can also factor $x^{21} - 1$ as

$$x^{21} - 1 = (x-1)(x^{20} + x^{19} + \cdots + 1) = (x-1)\sum_{j=0}^{20} x^j.$$

Comparing with the roots of unity factorization:

$$x^{21} - 1 = (x-1)\prod_{k=1}^{20}(x - \omega^k),$$

we conclude that

$$\prod_{k=1}^{20}(x - \omega^k) = \sum_{j=0}^{20} x^j.$$

Step 4: Evaluate at $x = 1$

Setting $x = 1$, we obtain

$$\prod_{k=1}^{20}(1 - \omega^k) = \sum_{j=0}^{20} 1^j = 21.$$

Answer

$$\boxed{21}$$

7. Let

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

be a complex polynomial of degree d, with $a_d \neq 0$. Prove that every complex root α of $f(x)$ satisfies

$$|\alpha| \leq \frac{\sum_{i=0}^{d-1} |a_i|}{|a_d|} \quad \text{or} \quad |\alpha| < 1.$$

Solution:

We begin by applying the triangle inequality in a suitable form.

For any complex number x,

$$|f(x)| = |a_d x^d + (a_{d-1}x^{d-1} + \cdots + a_0)|.$$

Using the reverse triangle inequality,

$$|f(x)| \geq |a_d x^d| - |a_{d-1}x^{d-1} + \cdots + a_0|.$$

Hence,

$$|f(x)| \geq |a_d||x|^d - \left|\sum_{i=0}^{d-1} a_i x^i\right|.$$

Now, since α is a root of $f(x)$, we have $f(\alpha) = 0$. Substituting $x = \alpha$,

$$0 = |f(\alpha)| \geq |a_d||\alpha|^d - \Big|\sum_{i=0}^{d-1} a_i \alpha^i\Big|.$$

Therefore,

$$|a_d||\alpha|^d \leq \Big|\sum_{i=0}^{d-1} a_i \alpha^i\Big|.$$

Applying the triangle inequality to the right-hand side,

$$|a_d||\alpha|^d \leq \sum_{i=0}^{d-1} |a_i|\,|\alpha|^i.$$

Case 1: $|\alpha| > 1$

For $|\alpha| > 1$, note that $|\alpha|^i \leq |\alpha|^{d-1}$ for all $i \leq d-1$. Thus,

$$|a_d||\alpha|^d \leq \sum_{i=0}^{d-1} |a_i|\,|\alpha|^{d-1}.$$

Dividing both sides by $|\alpha|^{d-1}$ (which is positive), we get

$$|a_d||\alpha| \leq \sum_{i=0}^{d-1} |a_i|.$$

Hence,

$$\boxed{|\alpha| \leq \frac{\sum_{i=0}^{d-1} |a_i|}{|a_d|}}.$$

Case 2: $|\alpha| \leq 1$

In this case, we simply have

$$\boxed{|\alpha| < 1.}$$

Conclusion.
Every complex root α of $f(x)$ satisfies

$$\boxed{|\alpha| \leq \frac{\sum_{i=0}^{d-1} |a_i|}{|a_d|} \quad \text{or} \quad |\alpha| < 1.}$$