Í OPENER

AN INTRO TO COMPLEX ANALYSIS

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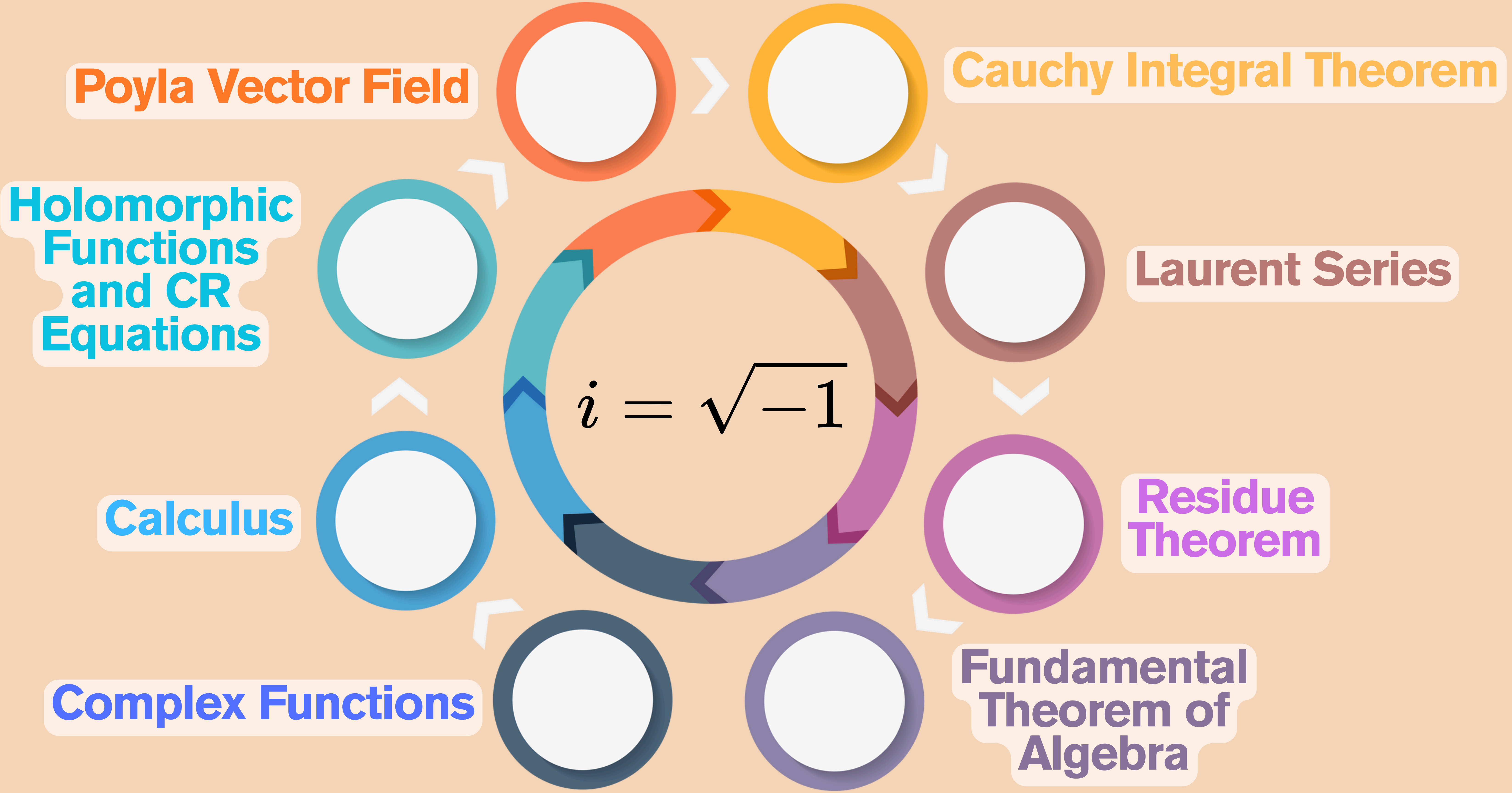
ROADMAP



Complex Functions
Calculus
Holomorphic Functions and CR Equations
Poyla Vector Field
Cauchy Integral Theorem
Laurent Series
Residue Theorem
Fundamental Theorem of Algebra
$i = \sqrt{-1}$

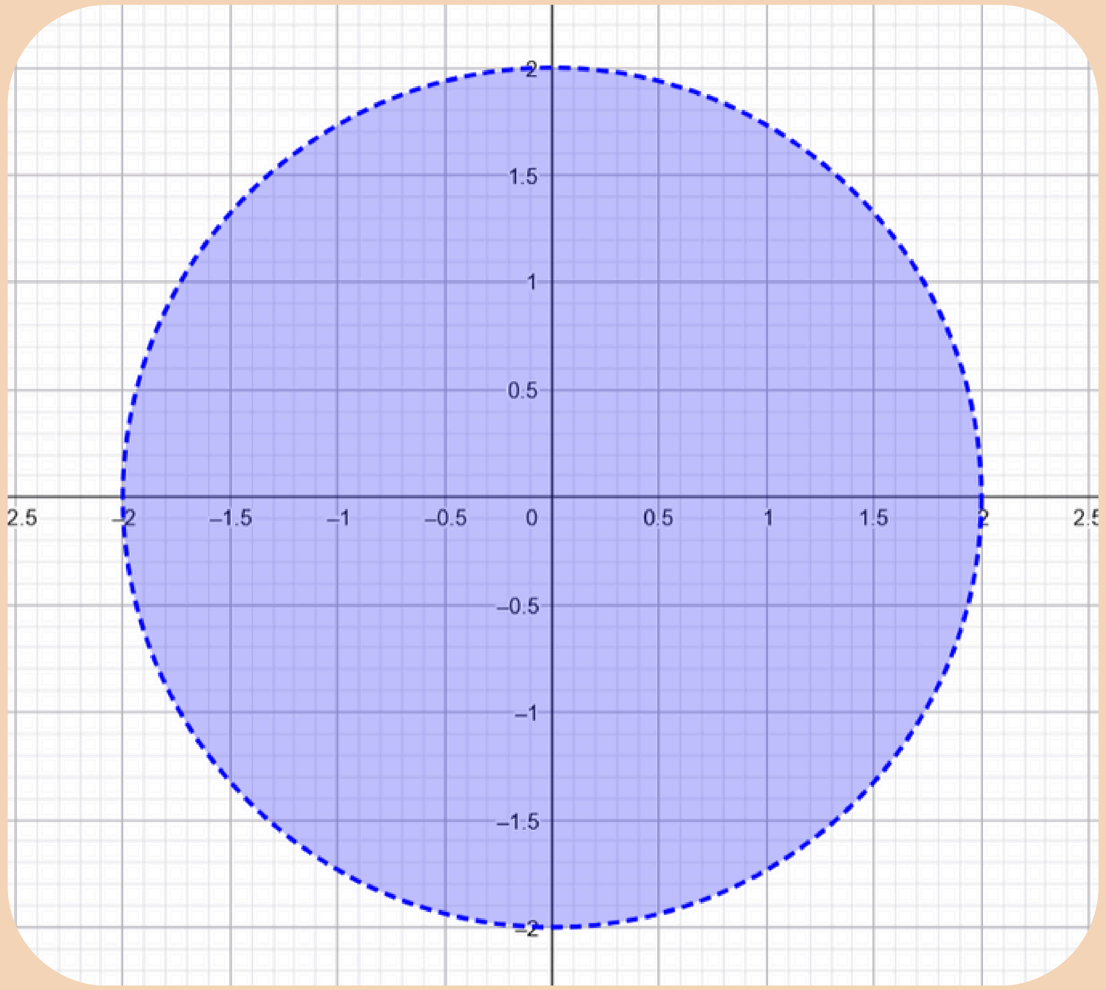
CONTINUITY

- How to write complex numbers differently??

$$f = u + iv;\ \text{where } u, v \text{ are real valued functions.}$$

- What are open and closed subsets?

CONTINUITY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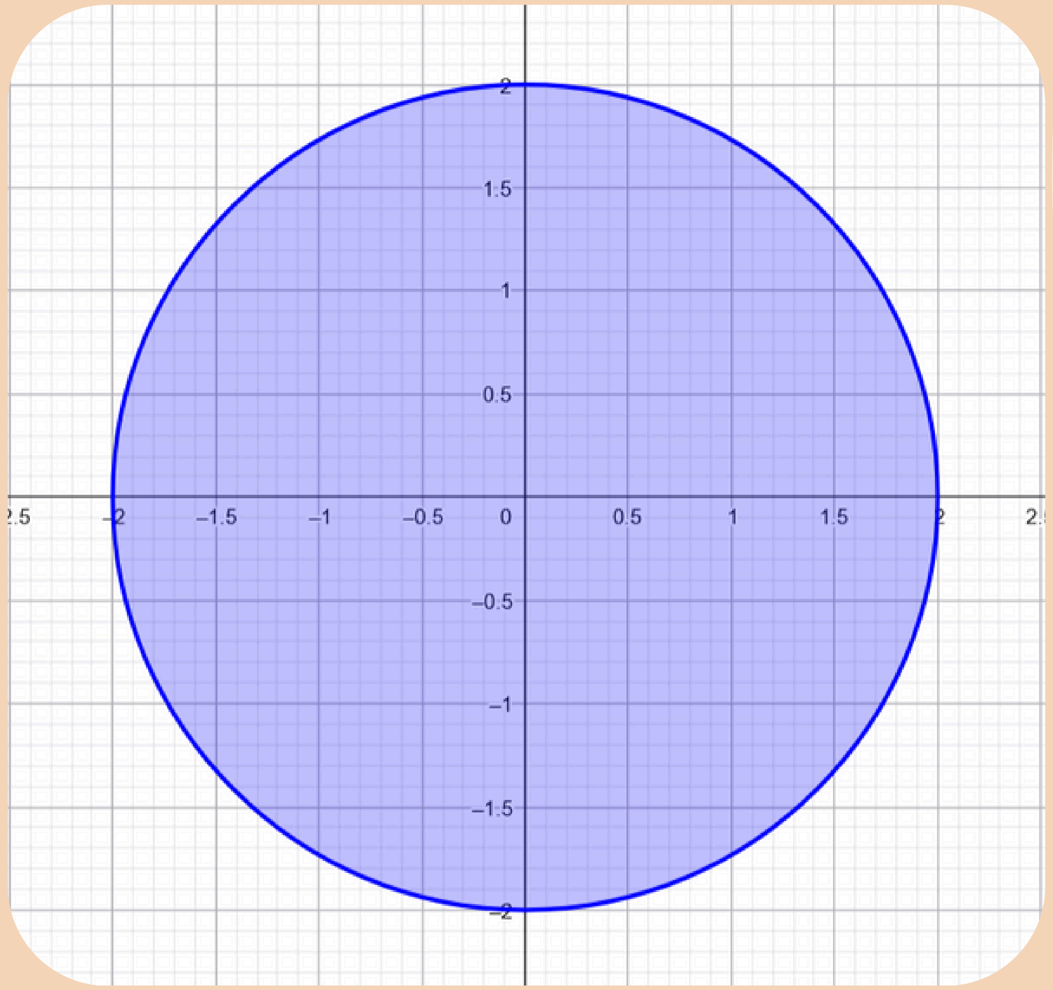
1.5
1
0.5
−2 −1.5 −1 −0.5 0 0.5 1 1.5 2
−0.5
−1
−1.5
−2


2
1.5
1
0.5
−2 −1.5 −1 −0.5 0 0.5 1 1.5 2
−0.5
−1
−1.5
−2

OKAY AND?

$f(z) = u + iv$ is continuous only if u, v
are continuous as real valued functions of two variables.

- **Take a minute to think if this is rigorous enough.**

DIFFERENTIABILITY

- How to write complex numbers differentiably??

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$z_n \in U \text{ with } \lim_{n \to \infty} z_n = z_0.$$

- This is the writing of a Mathematician, instead we'll look at Methematics.

HOLOMORPHIC FUNCTIONS

- Point z_0: If complex diff'able at all points in neighbourhood of z_0, holomorphic at z_0.
- Open Set U: If complex differentiable at all points, entire.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$z_n \in U \text{ with } \lim_{n \to \infty} z_n = z_0.$$

RIEMANN-CAUCHY EQUATIONS

- What is this? Other than baring the fancy name of two scientists.

$$f_x = -if_y.$$

- Is this a sufficient condition? If not, what can we add?

POLYA VECTOR FIELD

- Why? We'll look at a few integrals later. So, what kind of a mechanism do we need to compute them?

$$F = (u(x,y), -v(x,u)) = \bar{z}_{\text{coord.}}$$

$$W(z) = W(x,y) =< w_1(x,y), w_2(x,y) >$$

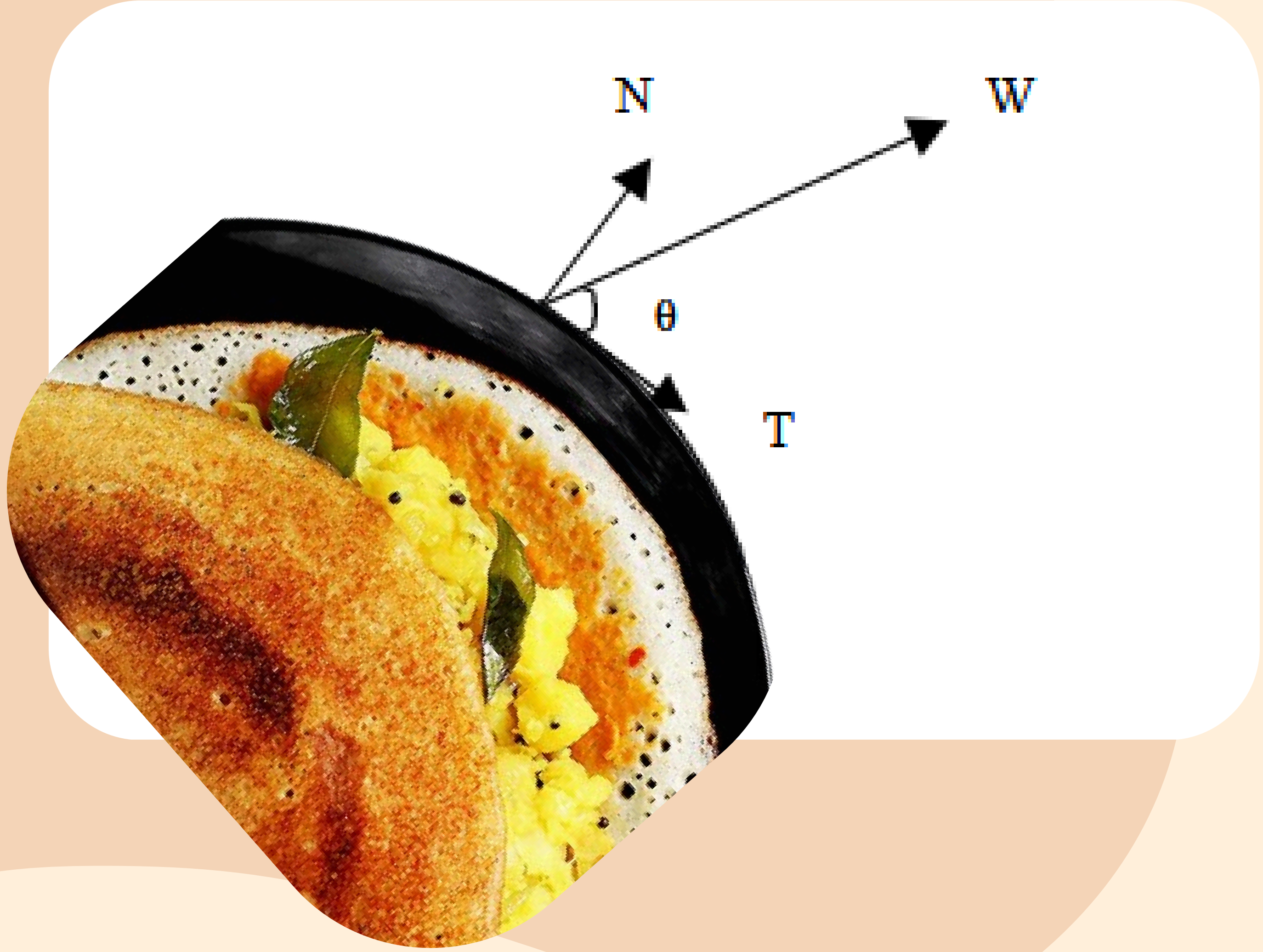
PALYA PYRAMID

$$\int_C f(z)dx$$

$$= \int_c (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i(udy + vdx)$$

$$= \int_C (w_1dx + w_2dy) + i(w_1dy - w_2dx)$$

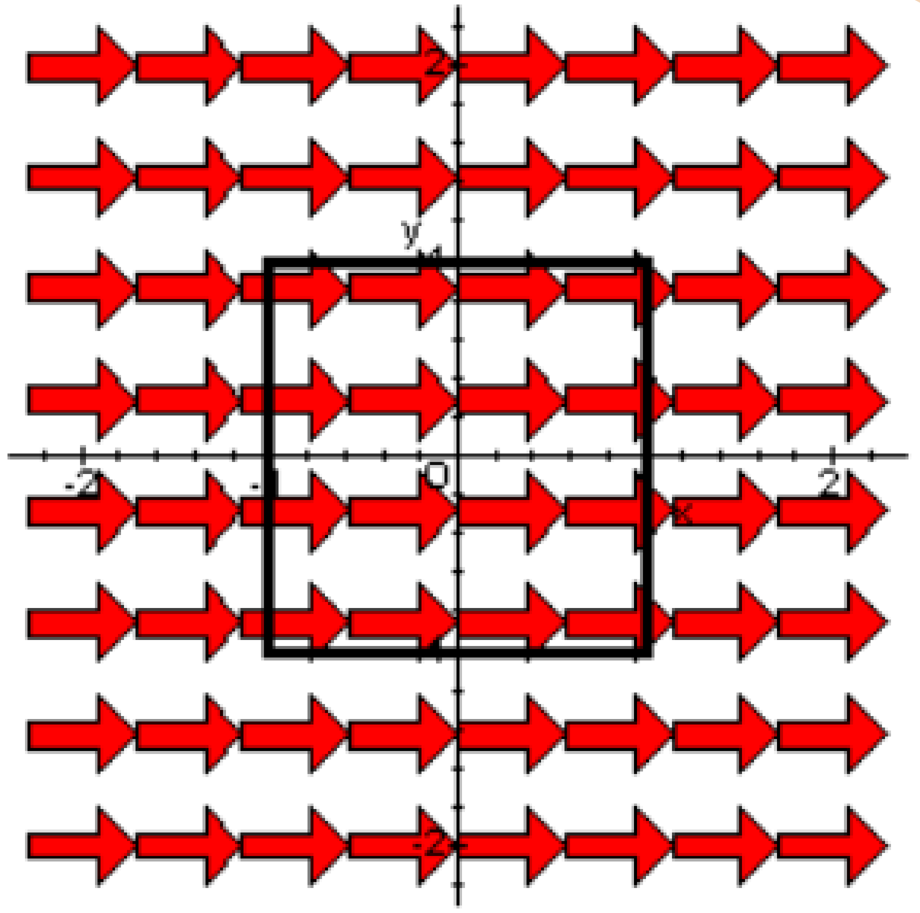


ALMOST THERE BRO

$$\int_C f(z)dz = \int_C W \cdot T dl + i \int_C W \cdot N dl$$

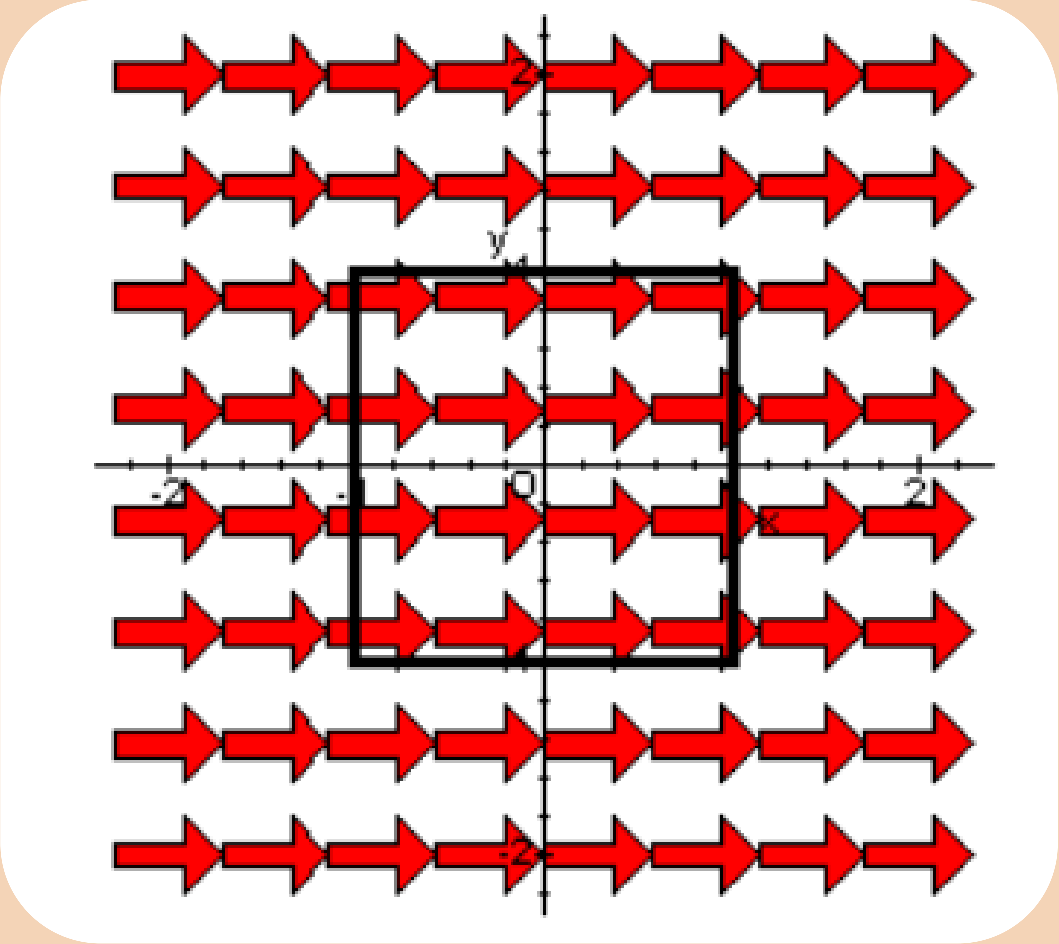
- The first integral represents the flow along the contour.
- Second integral represents flow perpendicular to contour. So essentially the flux through the contour.

WHEN I SLEEP IN CLASS FOR 1 SEC



$$\int_C f(z)dz = 0.$$

WHEN I SLEEP IN CLASS FOR 1 SEC + CONDITIONS FOR THIS

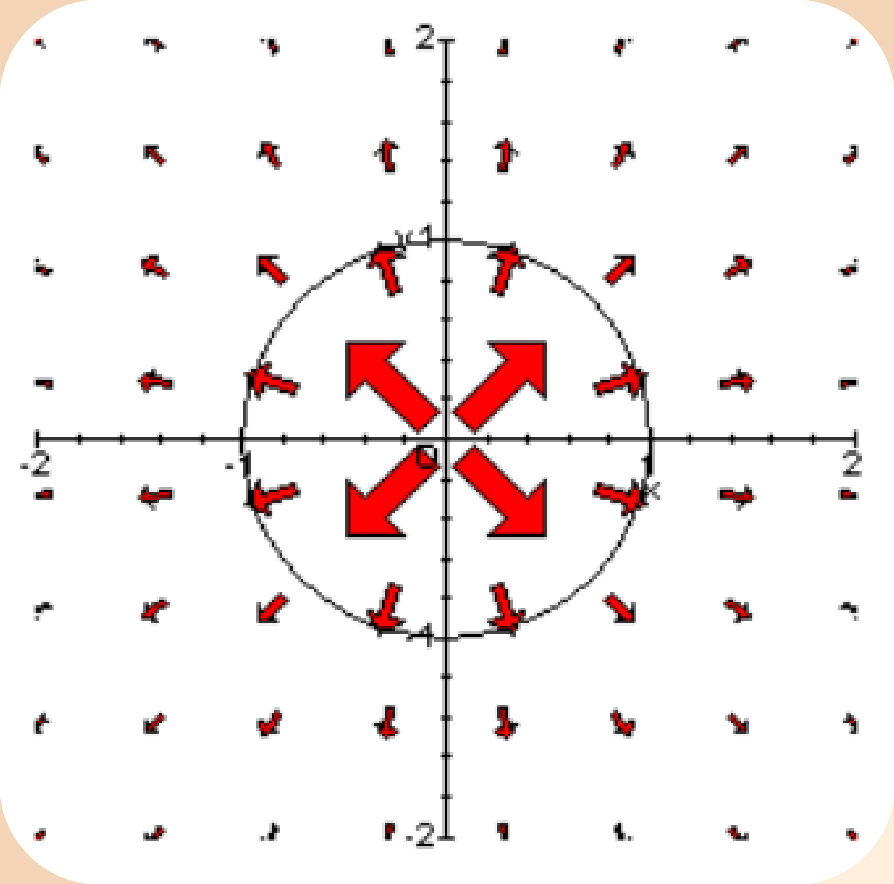


$$\int_C f(z)dz = 0.$$

When f is analytic in a simply connected domain D and C is a simple closed contour in D.

ONE SMOL ISUE

- What if there's a singularity inside the contour?



CAUCHY INTEGRAL THEOREM

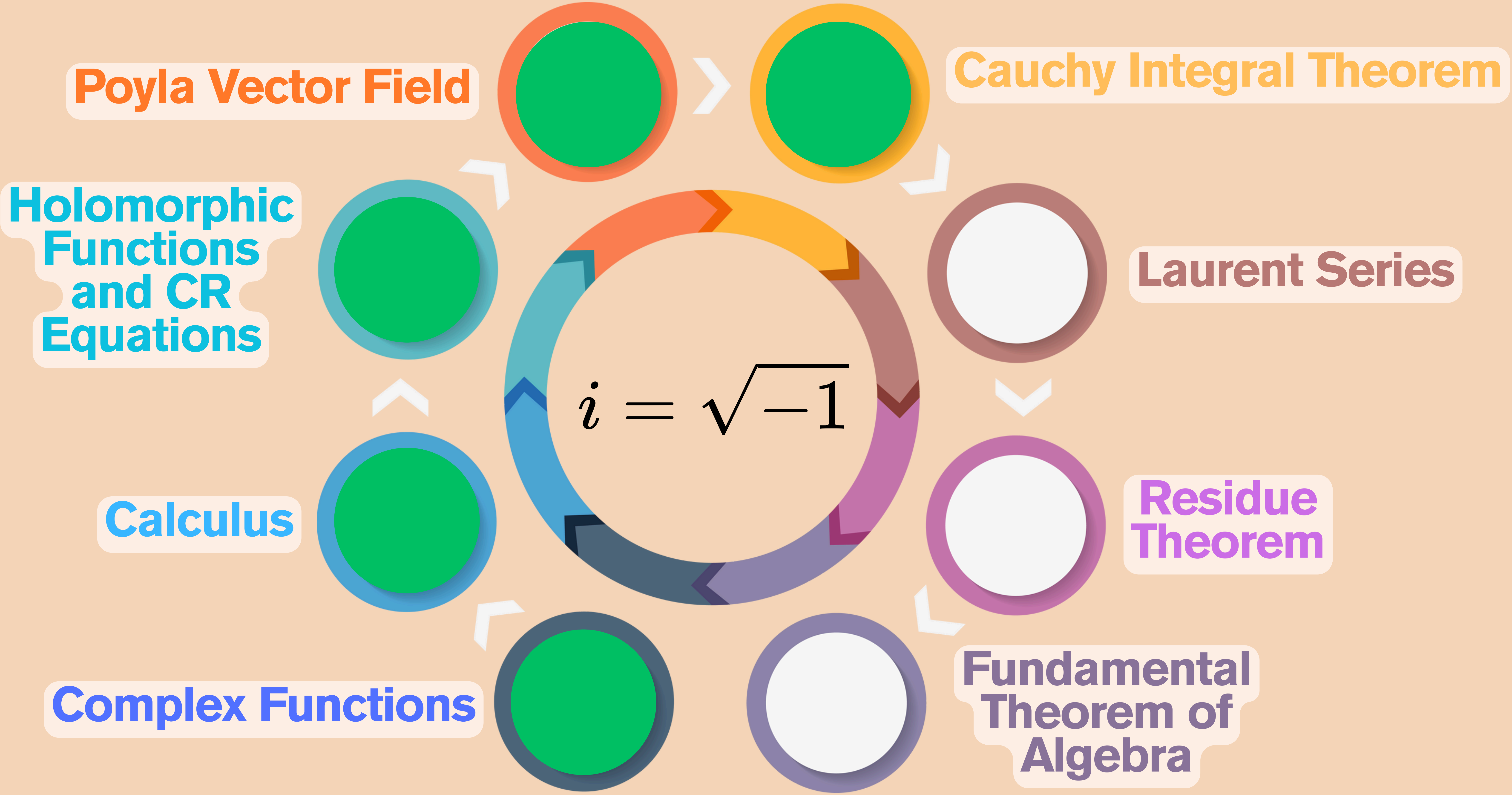
$$\oint_C \frac{1}{z-a}\,dz = 2\pi i,$$

counter clockwise integral

CAUCHY'S INTEGRAL FORMULA

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\,dz.$$

ROADMAP



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LAURENT SERIES

A LAURENT SERIES IS AN EXPANSION OF A COMPLEX FUNCTION F(Z) INTO AN INFINITE SERIES .

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

THE POSITIVE POWERS ARE ASSOCIATED WITH THE REGULAR PART OF THE SERIES . THE NEGATIVE POWERS REPRESENT THE SINGULARITIES OR THE POLES OF THE FUNCTION.

RESIDUES

A RESIDUE OF A COMPLEX FUNCTION F(Z) AT A POINT Z0 IS THE COEFFICIENT 1/(Z-Z0) IN THE LAURENT SERIES EXPANSION OF F(Z) AROUND Z0

$$Res(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}((z - z_0)^n f(z))$$

$$Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

RESIDUE THEOREM

THE RESIDUE THEOREM STATES THAT THE VALUE OF CONTOUR INTEGRALS OF ANALYTIC FUNCTIONS OVER CLOSED CURVES IS EQUAL TO 2ΠI TIMES THE SUM OF THE RESIDUES OF THE INTEGRAND AT THE SINGULARITIES ENCLOSED BY THE CONTOUR.

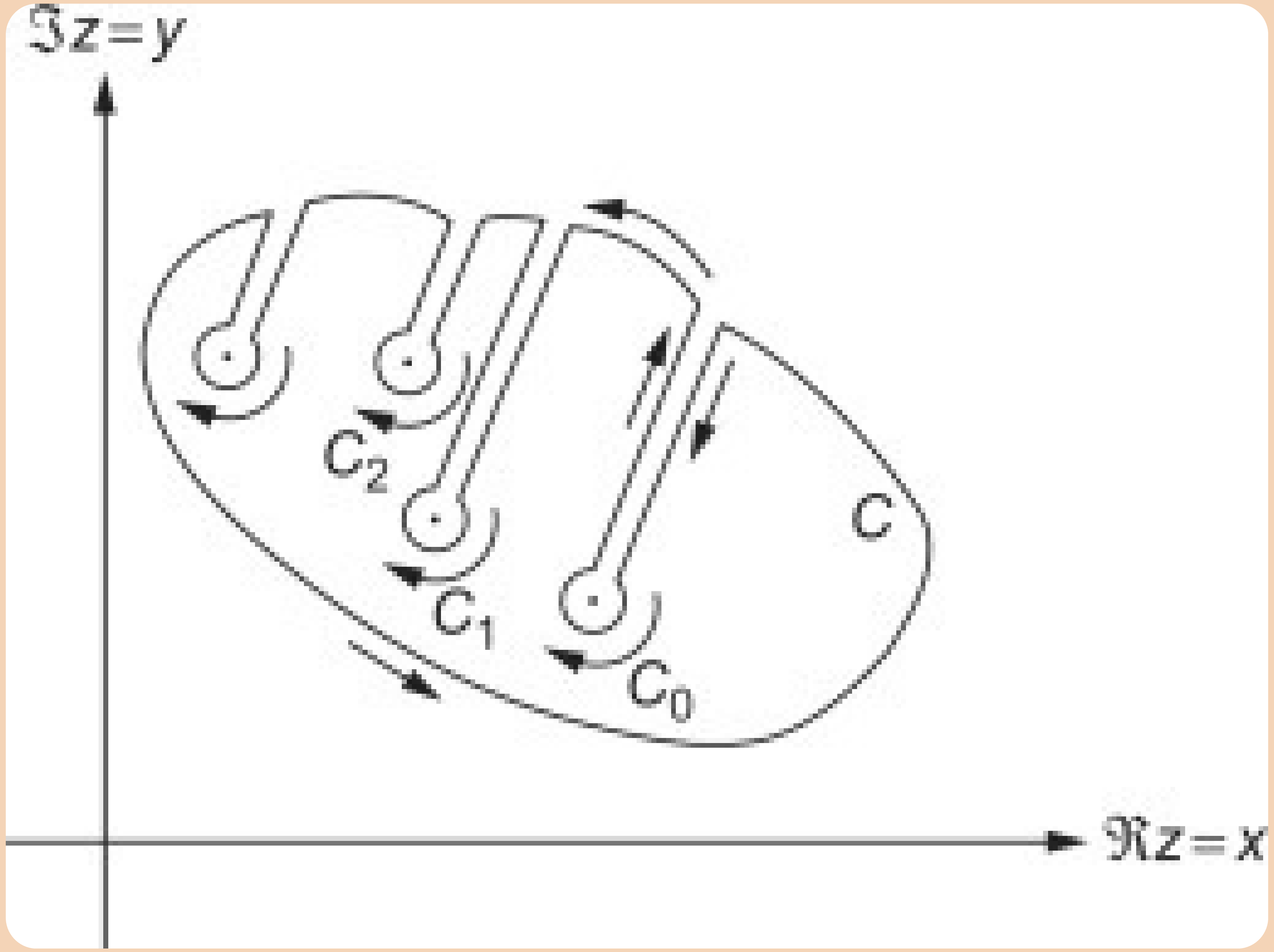
$$\oint_{\gamma} f(z)dz = 2\pi i \sum_{k=1}^{n} I(\gamma, a_k) Res(f, a_k)$$

RESIDUE THEOREM

OUTLINE OF THE PROOF: CONSTRUCT SMALL CONTOURS AROUND THE POLES AND REMOVE THEM FROM THE ORIGINAL CONTOUR. FROM CAUCHY'S THEORM THE INTEGRAL ALONG THIS NEW CONTOUR IS 0. HENCE

$$\oint_{\gamma} f(z)dz = \sum_{k=1}^{n} \oint_{\gamma_k} f(z)dz$$

RESIDUE THEOREM



RESIDUE THEOREM

NOW EXPAND THE LAURENT SERIES AROUND EVER POLE AND EVALUATE THE INTEGRAL

$$\oint_{|z|=1} \frac{1}{z} dz = 2\pi i$$

THIS INTEGRAL EVALUATES TO ZERO FOR ANY OTHER POWER OF Z AS ITS INTEGRAND.

RESIDUE THEOREM

THIS INTEGRAL EVALUATES TO ZERO FOR ANY OTHER POWER OF Z AS ITS INTEGRAND.

SO THE RHS EVALUATES TO A CONSTANT TIMES SUM RESIDUES AT ALL THE POLES

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} I(\gamma, a_k) Res(f, a_k)$$

TRY THIS INTEGRAL
IF YOU GET IT KK WILL TREAT YOU AND ME

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2} dx$$

THE FUNDAMENTAL THEOREM OF ALGEBRA

THE FUNDAMENTAL THEOREM OF ALGEBRA STATES THAT ANY NON-CONSTANT POLYNOMIAL WITH COMPLEX COEFFICIENTS HAS AT LEAST ONE COMPLEX ROOT.

THE FUNDAMENTAL THEOREM OF ALGEBRA

ROUGH OUTLINE OF THE PROOF:

LIOUVILLE'S THEOREM:
ANY BOUNDED ENTIRE FUNCTION MUST BE CONSTANT.

THE FUNDAMENTAL THEOREM OF ALGEBRA

SUPPOSE P(Z) HAS NO ROOTS IN C. THEN 1/P(Z) IS ENTIRE (ANALYTIC EVERYWHERE), SINCE P(Z)≠0 EVERYWHERE.
NEXT, ANALYZE THE BEHAVIOR OF 1/P(Z) AS ∣ Z ∣ →∞
IT HAS TO DIE DOWN TO ZERO.

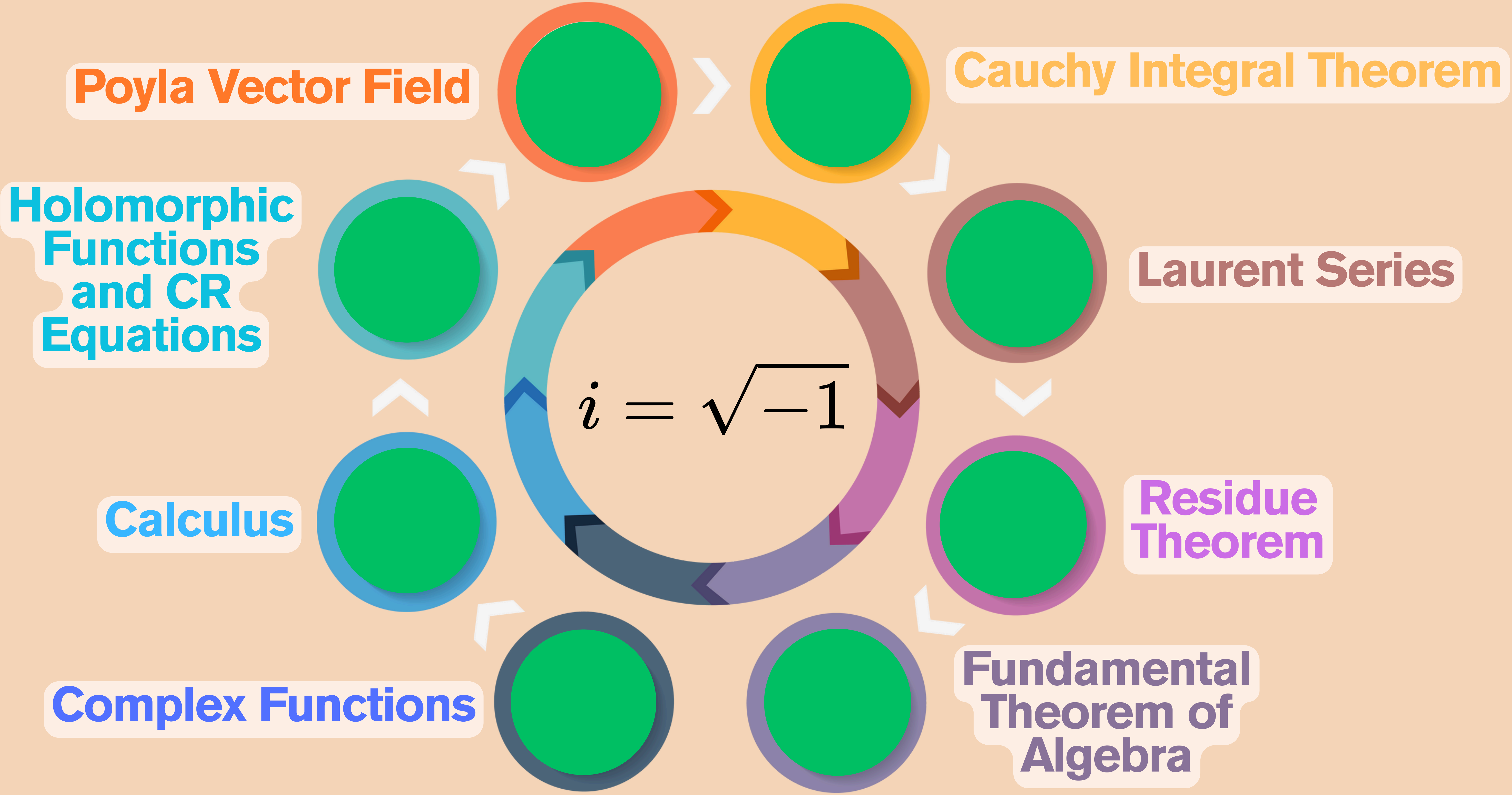
THE FUNDAMENTAL THEOREM OF ALGEBRA

THIS MEANS 1/P(Z) IS BOUNDED AT INFINITY, AND BY CONTINUITY, IT'S BOUNDED EVERYWHERE IN C.
BY LIOUVILLE'S THEOREM, 1/P(Z) IS CONSTANT, SO P(Z) IS CONSTANT - CONTRADICTING OUR ASSUMPTION.

THE FUNDAMENTAL THEOREM OF ALGEBRA

HENCE, P(Z) MUST HAVE AT LEAST ONE ROOT.
NOW WE CAN PROCEED BY DECREASING THE ORDER OF THE POLYNOMIAL UNTIL WE GET A CONSTANT TO FINALLY SHOW THAT THERE ARE N ROOTS.

ROADMAP



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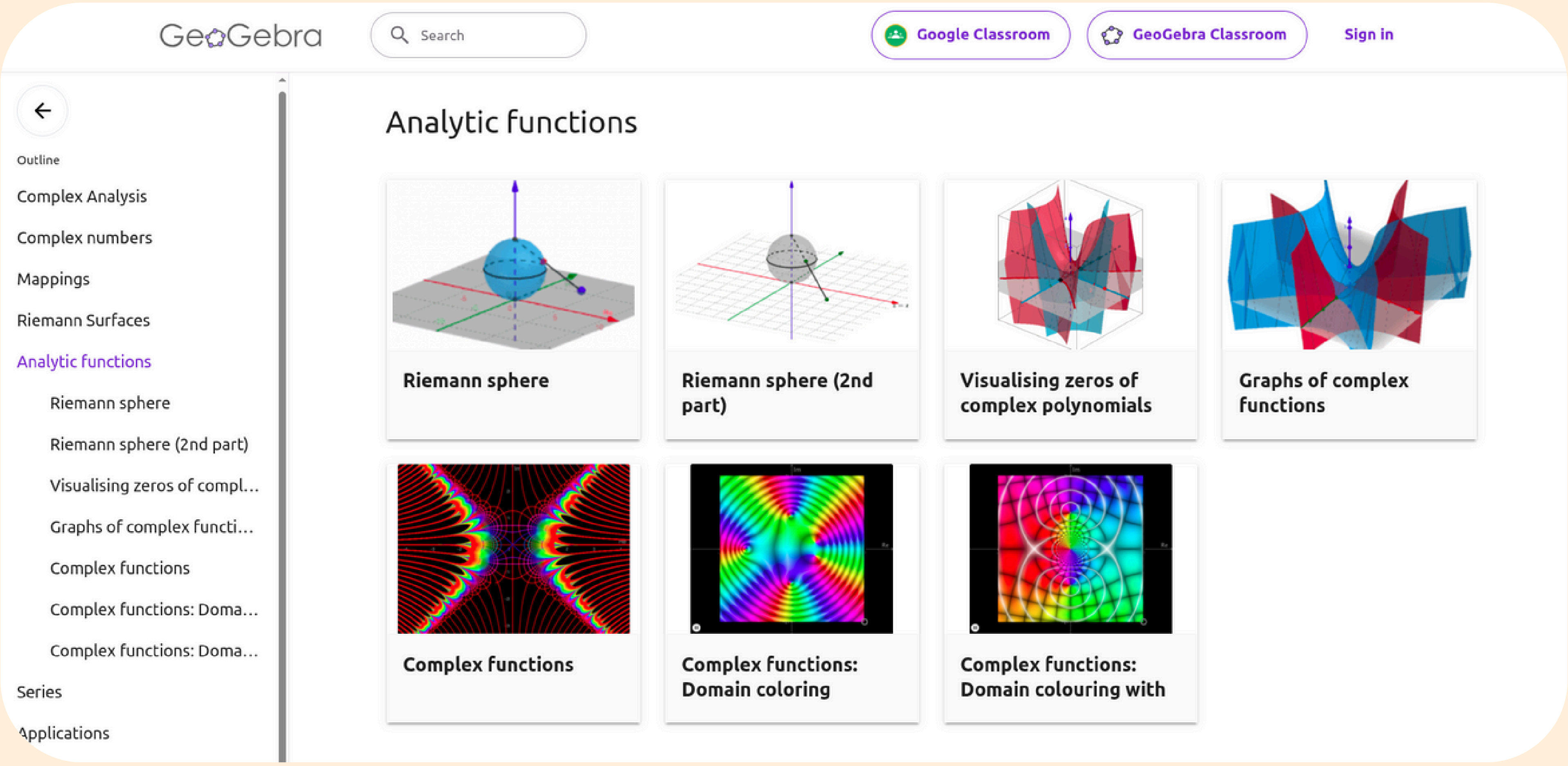
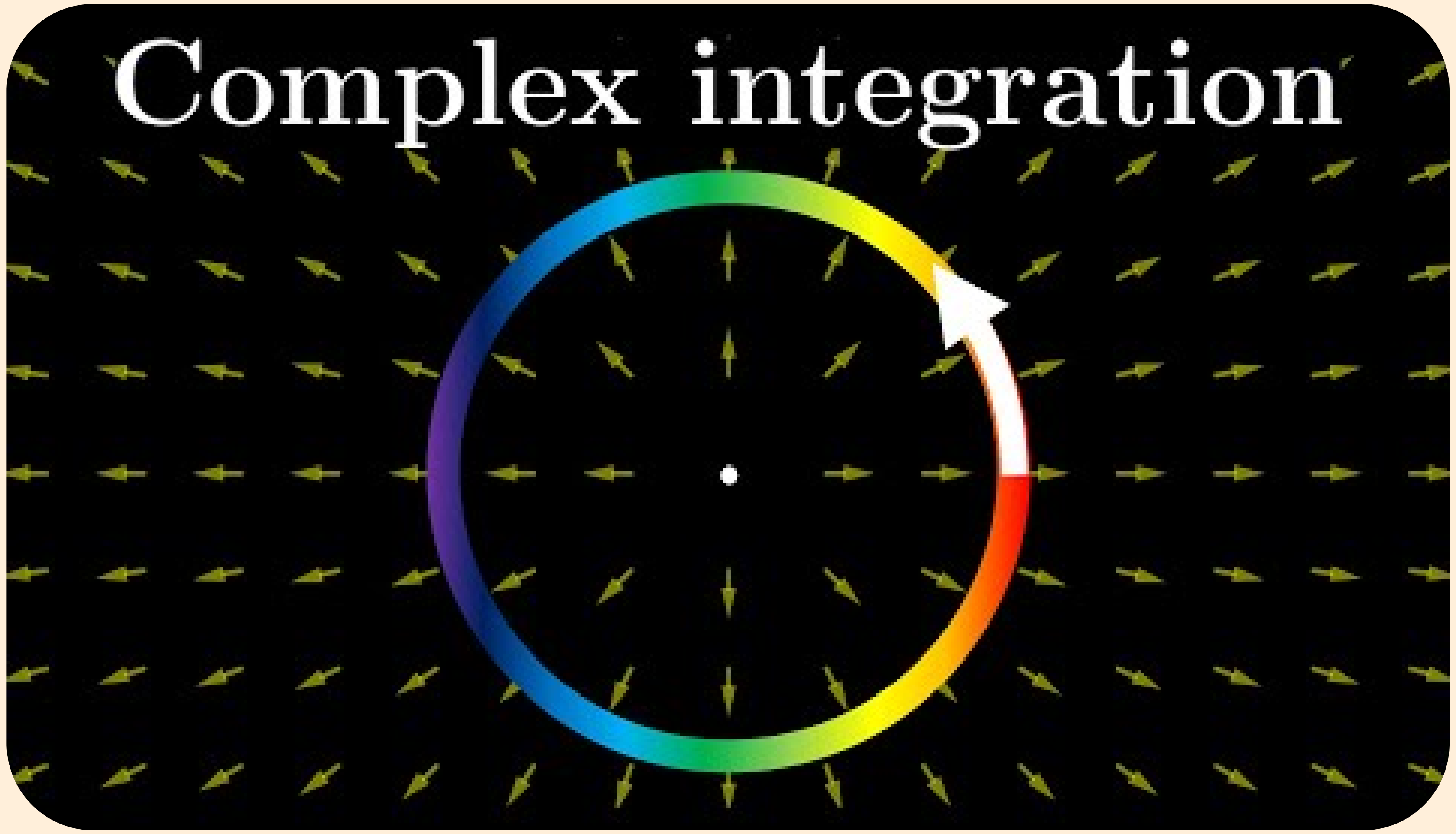
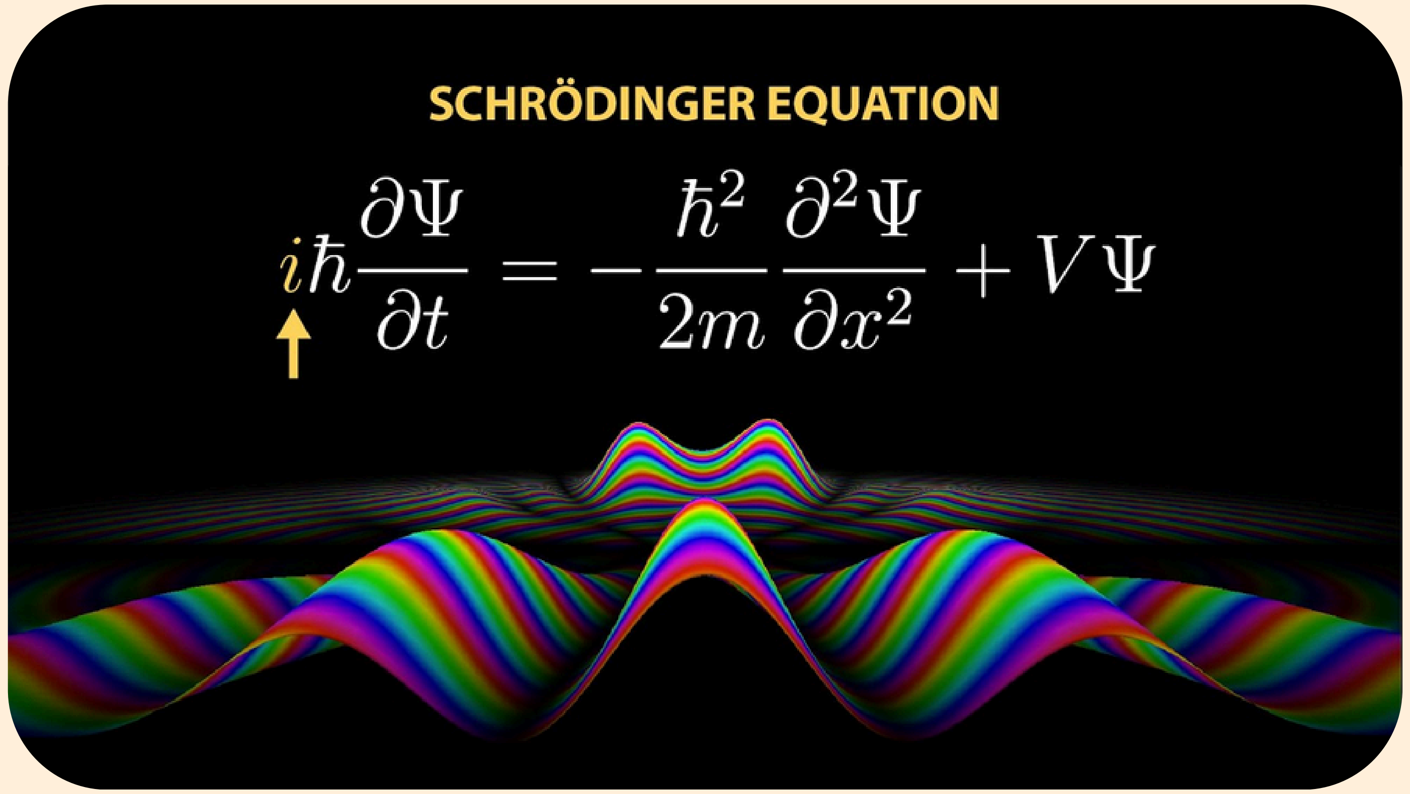

GeoGebra
Search
Google Classroom
GeoGebra Classroom
Sign in
Outline
Complex Analysis
Complex numbers
Mappings
Riemann Surfaces
Analytic functions
Riemann sphere
Riemann sphere (2nd part)
Visualising zeros of compl...
Graphs of complex functi...
Complex functions
Complex functions: Doma...
Complex functions: Doma...
Series
Applications
Analytic functions
Riemann sphere
Riemann sphere (2nd part)
Visualising zeros of complex polynomials
Graphs of complex functions
Complex functions
Complex functions: Domain coloring
Complex functions: Domain colouring with


Complex integration


SCHRÖDINGER EQUATION
$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$

THANK YOU