MathLab: Problem Set 3

Mathematics Club IITM

Navin, KK, Achintya

June 2024

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*For those with 7/10 attendance in the Forms, we will grant you attendance and qualify you in the attendance criteria required for the certificate, provided you solve **AT LEAST 2 OUT OF THE 3 BONUS QUESTIONS** correctly in addition to the rest of the assignment. The bonus questions are completely optional for those with at least 8/10 attendance but is mandatory for those with 7/10 in order to obtain a certificate.*

Instructions

- This problem set consists of 3 sections and you need to solve **all 3 sections to complete this course**.
- You are supposed to write your answers and upload it in the submission link given. You can submit a scanned copy of handwritten answers, submit a typed document or a handwritten document on a tablet. Anything works as long as it is legible and clear!
- This assignment requires you to write code, generate plots and submit the same. You are advised to submit the code and the plots by **attaching them in your submission document directly**. If that is not possible then please upload your code and plots to your Google Drive / GitHub and add the links to those files in your submission. **Make sure to give us access to those files.**
- Try to answer all the questions in a clear and readable manner and mention all your assumptions/reasons explicitly.
- It is fine even if you aren't able to solve the question completely after your best attempt. But **show us your working** or thought-process and the attempts that you have made in order **to clear the course**.
- The deadline for submission is 24 June 2024, 11:59 p.m.
- Feel free to reach out to us for doubts! Contact information of the problem-set creators:

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Section 1 Section 2 Section 3 Appendix

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§1 The Basics of Manipulation

This section contains simple conceptual-level questions which test your understanding of the topic and prepare you for the upcoming sections. **Show your working in each question.**

1. *e^x and a constant were walking down the street one day. While e^x continued walking, oh dear had the constant run away. What operator in the path would cause the constant much dismay?*

   Find the solutions $y(x)$ to the following differential equations given the boundary conditions:

   a) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 3y = 0$
   [$y(2) = e^{-2} + e^{12}$ and $y(16) = e^{-30} + e^{5}$]

   Final Expected Answer: y(6)

   b) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 3y = 15\sqrt{2}e^{ix}$
   [Assume only the forced part of the solution as discussed in the presentation i.e. system is at rest in the beginning]

   Final Expected Answer: The magnitude of the solution (Since the solution will be complex)

   c) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 5y = \sum_{k=-\infty}^{\infty} 25\sqrt{13}k^2 e^{\frac{kix}{2}}$
   [Assume only the forced part of the solution as discussed in the presentation i.e. system is at rest in the beginning]

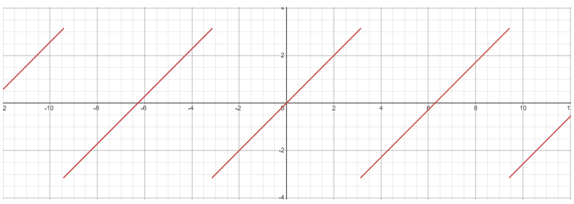
   Final Expected Answer: The magnitude of the coefficient of the $k = 2$ term in the solution series.

2. *Why did the function f go see a Fourier therapist? Because it wanted to break down its complex personality into a series of sines and cosines!*

   Consider the function:

$$f(x) = x, -\pi \leq x < \pi$$

   Here f has a period of 2π, that is $f(x + 2\pi) = f(x)$. Find the Fourier series corresponding to $f(x)$.



A tiny plot of the periodic function $f(x)$

Final Expected Answer: If the coefficients of the series are a_n, b_n, then find the value of $a_2 + b_2$.

§2 The Hunt for Harmonics

§2.1 A janky spring

You are expected to use the Fourier series to solve this problem. Doing so requires that you have knowledge of both the exponential and trigonometric series. However, in order to help you out, we have provided the relevant conversion formulae to convert a given Fourier trigonometric series to an exponential series and vice versa in the appendix. We highly encourage you work them out for yourselves.

Achintya remembers learning about an interesting function called the ramp function $r(t)$ in his signal processing class. He remembers his teacher defining it as:

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & otherwise \end{cases}$$

On his class quiz, he comes across this differential equation when he attempts to model an old, rusty and stiff spring that is being slowly but steadily pulled by an insanely strong ant. However since he missed half of his math classes, he has no idea how to solve it. Can you help him?

$$5\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x = r_p(t)$$

Assume $r_p(t) = r(t)$ when $-\pi < t \leq \pi$ and is periodic with time period 2π. Also solve for only the particular solution (this is what we discussed in the session).

The insanely strong ant

Final Expected Answer: The solution to the differential equation given above using Fourier series methods.

Bonus Question 1: A recovering Ant [For those with 7/10 attendance]:

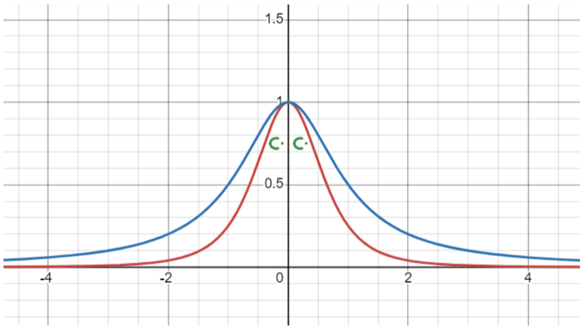
Consider the function $f(t) = |t|$ when $-\pi < t \leq \pi$ with the same period as the above question ($T = 2\pi$). Can you relate it somehow to the ramp function? Now use that knowledge to solve the above question but with the driving force replaced by $f(t)$ (with the same period) instead of the ramp function.

NOTE: *For those who have already met the attendance criteria, there is absolutely* ***NO*** *need to submit this bonus question. But I still ask you to try this out. Including it in your solution will make my day.*

§2.2 Hmm, integrals

The integral $\int \frac{1}{1+x^2} dx$ is something that you would have encountered a lot in your high school calculus and some of you might even scream tan in verse ex plus sea (yes, live with it) the moment that you see this fairly innocent integral.

But if I were to task you to find $\int \frac{1}{(1+x^2)^2} dx$ you would be walking into integration-by-parts and partial-fractions war-zone territory. Let us not ponder over that mess now!



A spooky figure made using $\frac{1}{1+x^2}$ *and* $\frac{1}{(1+x^2)^2}$

If I were to simplify things a bit, and ask you to find the value of the definite integral

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$$

what would your approach be? There is a really elegant way to compute this integral using the concepts that were taught in the third session. Try to recall them and compute the value of this definite integral.

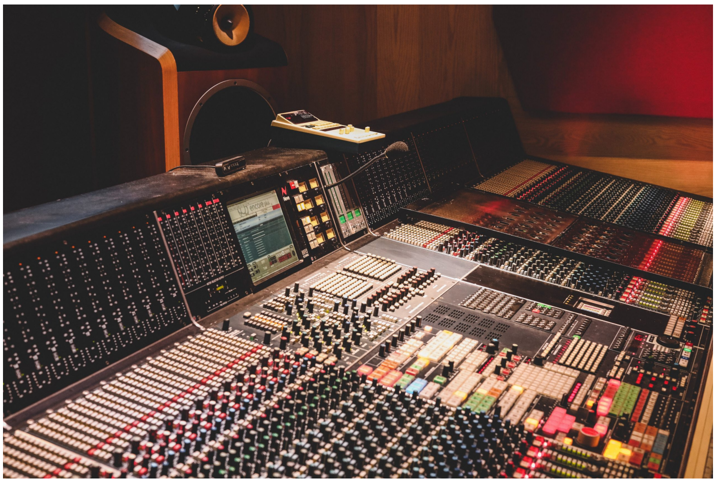
Hint: Think of a function whose Fourier transform is $\frac{1}{1+x^2}$ *or in other terms finding the inverse Fourier transform of* $\frac{1}{1+x^2}$ *might be helpful to you*

Final Expected Answer: The value of the definite integral $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$ computed using the concepts taught in the third session.

§3 A Tale of Musical Mismanagement

Welcome to the clumsy life of Navin!

This morning Navin was assigned the work of recording an orchestral performance that was being performed by the coordinators of the Mathematics Club. He was asked to record each instrument in a different channel, so that the audio can be equalized easily later on by our Music Directors - KK and Achintya. Unfortunately, Navin being the clumsy person that he is accidentally ended up recording all of the instruments *as a single audio file*.



The Music directors want to **tone down** (decrease the volume of) **the snare drums by 25%**. The snare drums used by the Mathematics Club produce sounds in the **frequency range of 150Hz to 250Hz**. Assume that no other musical instruments produce sounds in this audio range.
Luckily, he has *you* to help him! Navin hoping that you listened to his presentation attentively is asking you to write a program in `MATLAB` or `GNU Octave` which fulfills the directors' request and saves his job!

Final Expected Submission: A script written in `MATLAB` or `GNU Octave` that performs the following:

- Reads an audio file named *"i_hope_i_don't_lose_my_job.wav"* using the function `audioread` as an array along with the sample-rate of the audio
- Uses Fourier Transform methods to perform the request of reducing the volume of the drums by 25% in the audio data
- Writes the newly generated audio data using the function `audiowrite` to an audio file named *"thankyou_for_saving_my_job.wav"*

Note:

1. The functions `audioread`, `audiowrite` have been documented here: 1, 2, 3.
   Feel free to reach out to us in case you don't understand how these functions work!
2. If you are submitting your code via a Google Drive / GitHub link make sure to enable access for all, so that we will be able to access your submission.

Bonus Question 2: Thinking in the Time Domain [For those with 7/10 attendance]:

NOTE: *For those who have already met the attendance criteria, you DO NOT need to submit this! You will be eligible for the certificate even if you don't attempt this question. This is just a bonus question. I will be very happy if you attempt this question, and that is a good thing!*

Can you think of a way to perform the same task of changing the contribution of some frequencies in a given function of time $f(t)$ **without** analysing the function in the frequency-domain using a Fourier Transform, instead working entirely in the time-domain. You can use the same example as the above question or use any example that you like.

Explain your algorithm.

Bonus Question 3: A more sophisticated Filter [For those with 7/10 attendance]:

NOTE: *For those who have already met the attendance criteria, you DO NOT need to submit this! You will be eligible for the certificate even if you don't attempt this question. This is just a bonus question. I will be very happy if you attempt this question, and that is a good thing!*

Repeat Problem 3 (A Tale of Musical Mismanagement) but instead of toning down the amplitude of all frequencies in 150Hz to 250Hz by 25%, you have to tone down the frequencies:

- 150Hz to 175Hz by 20%,
- frequencies 175Hz to 225Hz by 25%,
- and frequencies 225Hz to 250Hz by 15%.

Follow the same instructions as given in Problem 3.

Happy Solving!

Appendix

Let the exponential series be of the form $\sum_{k=-\infty}^{\infty} A_k e^{ik\omega_0 t}$ and the trigonometric series be of the form $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k cos(\omega_0 t) + \sum_{k=1}^{\infty} b_k sin(\omega_0 t)$. Then, it can be shown that:

- $a_k = A_k + A_{-k}$
- $b_k = i(A_k - A_{-k})$

Note that $k \in \mathbb{Z}, k \geq 0$ or put simply k ranges from 0 (inclusive) to ∞ in integers.