Mathematics Guild
Solutions to Problem Set - 3

Challenge posed on: 08/07/2025 **Challenge conquered by: 14/07/2025**

1 Overview

- **Topics focused:**
  - Algebra
  - Polynomials
  - Functional Equations
- **Challengers:**
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- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Happy solving :)

2 Problems

1. **Functional Inequality on Natural Numbers** Let $f : \mathbb{N} \to \mathbb{N}$ satisfy

$$2n + 2024 \le f\big(f(n)\big) + f(n) \le 2n + 2026 \quad \text{for every } n \in \mathbb{N}. \tag{1}$$

1. Setting up the auxiliary sequences Fix an arbitrary $n \in \mathbb{N}$ and define

$$a_0 = n, \qquad a_{k+1} = f(a_k) \quad (k \ge 0),$$

together with the "error" terms

$$c_k = a_{k+1} - a_k - 675 \qquad (k \ge 0).$$

By construction $a_{k+1} = a_k + 675 + c_k$.

Insert these expressions into (1):

$$a_{k+2} + a_{k+1} = \big(a_k + 1350 + c_k + c_{k+1}\big) + \big(a_k + 675 + c_k\big) = 2a_k + 2025 + 2c_k + c_{k+1}.$$

Hence

$$2a_k + 2024 \le 2a_k + 2025 + 2c_k + c_{k+1} \le 2a_k + 2026 \Longrightarrow -1 \le 2c_k + c_{k+1} \le 1. \tag{2}$$

2. A useful sign-growth dichotomy Rewrite (2) as

$$c_{k+1} \le 1 - 2c_k, \qquad c_{k+1} \ge -1 - 2c_k. \tag{3}$$

- If $c_k \ge 1$, then the first inequality in (3) forces $c_{k+1} \le -1$.
- If $c_k \le -1$, the second inequality in (3) gives $c_{k+1} \ge 1$.

Thus *whenever* $c_k \neq 0$ *the signs of consecutive* c*-terms alternate and their absolute values never drop below* 1.

3. Assuming $c_0 \neq 0$

Case A: $c_0 \geq 1$. We prove by induction that

$$c_{2k} \ \geq \ k+1 \qquad (k \geq 0). \tag{4}$$

Base $k=0$. $c_0 \geq 1 = 0+1$.
Induction step. Assume $c_{2k} \geq k+1$.
From $2c_{2k} + c_{2k+1} \leq 1$ we get $c_{2k+1} \leq 1 - 2c_{2k} \leq -2k-1$.
Plugging this into $-1 \leq 2c_{2k+1} + c_{2k+2}$ yields $c_{2k+2} \geq -1 - 2c_{2k+1} \geq 4k+1 \geq k+2$ for every $k \geq 0$.
Hence (4) is established.

With (4) we bound the original sequence. From

$$a_{k+1} = a_k + 675 + c_k, \quad a_{k+2} = a_{k+1} + 675 + c_{k+1}$$

we obtain

$$a_{2k+2} = a_{2k} + 1350 + c_{2k} + c_{2k+1}.$$

Using $c_{2k+1} \leq 1 - 2c_{2k}$ we deduce

$$a_{2k+2} \leq a_{2k} + 1351 - c_{2k} \leq a_{2k} + 1350 - k \quad \text{(by (4))}. \tag{5}$$

Iterating (5) gives

$$a_{2k} \leq a_0 + \sum_{j=0}^{k-1} (1350 - j) = a_0 + 1350k - \frac{k(k-1)}{2}. \tag{6}$$

The quadratic term $-k(k-1)/2$ dominates the linear one, so (6) becomes negative for sufficiently large k, contradicting $a_{2k} \in \mathbb{N}$. Thus $c_0 \ngeq 1$.

Case B: $c_0 \leq -1$. Now $2c_0 + c_1 \geq -1$ implies $c_1 \geq 1$.
Starting the previous argument at index 1 (i.e. with the pair c_1, c_2) again leads to a contradiction.

Therefore $c_0 = 0$.

4. Concluding the form of f Because $c_0 = 0$,

$$a_1 = a_0 + 675 \quad \Longrightarrow \quad f(n) = n + 675 \qquad (n = a_0).$$

Finally,

$$f\big(f(n)\big) + f(n) = (n+1350) + (n+675) = 2n + 2025,$$

which indeed lies between $2n+2024$ and $2n+2026$. Hence

$$\boxed{f(n) = n + 675 \text{ for all } n \in \mathbb{N}}$$

and the solution is unique.

2. **Too cool to be in GP** Let $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, where $a_3, a_2, a_1, a_0 \in \mathbb{Z}$.
Assume that the roots of the polynomial form a geometric progression with ratio q, where $|q| \neq 1$ and $q \neq 0$.
Let the roots be a, aq, aq^2, where $a \in \mathbb{R}$ and irrational.

By Vieta's formulas, the sum of the roots satisfies:

$$a + aq + aq^2 = a(1 + q + q^2) = -\frac{a_2}{a_3}$$

and the sum of the products of the roots taken two at a time is:

$$a \cdot aq + a \cdot aq^2 + aq \cdot aq^2 = a^2q(1 + q + q^2) = \frac{a_1}{a_3}$$

Dividing the second equation by the first:

$$\frac{a^2q(1 + q + q^2)}{a(1 + q + q^2)} = aq = \frac{a_1}{-a_2}$$

Hence,

$$aq = \frac{a_1}{-a_2}$$

which is rational, since $a_1, a_2 \in \mathbb{Z}$ and $a_2 \neq 0$.
But this implies that one of the roots, aq, is rational, contradicting the assumption that all roots are irrational.

Therefore, the roots of a cubic polynomial with integer coefficients cannot be three positive irrational numbers in geometric progression.

3. **This is more confusing than Pokemon XYZ** Given: $f : R \to R$

$$x(f(x + y) - f(x - y)) - y(f(x + y) + f(x - y)) = 8xy(x^4 - y^4)$$

With, f(1) = 2025. The passkey on the first day of the month.

The key trick in this problem is that: if x and y are independent variables, then x + y and x - y are also independent. That is, it is possible to choose any two real numbers a and b, with x + y = a and x - y = b. So, put a = x + y, b = x - y and solve for f(a) and f(b) where a and b can be varied independent of each other.

First we simplify the equation, grouping the f(x + y) and f(x - y) terms:

$$\begin{aligned} f(x + y) * (x - y) - f(x - y) * (x + y) &= 8xy(x^2 + y^2)(x^2 - y^2) \\ &= 8xy(x^2 + y^2)(x + y)(x - y) \\ &= 4xy * (2x^2 + 2y^2) * (x + y)(x - y) \\ &= [(x + y)^2 - (x - y)^2] * [(x + y)^2 + (x - y)^2] * (x + y)(x - y) \end{aligned}$$

Making the substituion: $a = x + y, b = x - y$, we get:

$$bf(a) - af(b) = (a^2 + b^2) * (a^2 - b^2) * ab = ab * (a^4 - b^4)$$

Dividing by ab, we get:

$$\frac{f(a)}{a} - \frac{f(b)}{b} = a^4 - b^4 \Rightarrow \frac{f(a)}{a} - a^4 = \frac{f(b)}{b} - b^4 = c$$

for some constant c.

Thus, $f(a) = a * (c + a^4) = a^5 + ca$. Since $f(1) = 2025$, we get $f(1) = 1 + c = 2025 \Rightarrow c = 2024$.
Therefore, $f(x) = x^5 + 2024x$.

4. **Even more confusing than Pokemon XYZ** Given: $f : \mathbb{R} \to \mathbb{R}$

$$f(xy + f(x)) + f(y) = xf(y) + f(x + y)$$

for all real numbers x and y.

Let $P(x, y)$ be the assertion $f(xy + f(x)) + f(x + y) = xf(y) + f(x + y)$.
$P(0, 0) \implies f(f(0)) + f(0) = 0 + f(0) \implies f(f(0)) = 0$
$P(0, 0), 0 \implies 2f(0) = f(0)^2$ and so $f(0) \in \{0, 2\}$

Case i: If $f(0) = 2$ and so $f(2) = f(f(0)) = 0$
$P(x, 0) \implies f(f(x)) + 2 = 2x + f(x)$ and so $f(x)$ is injective.
For a rigorous proof: Assume $f(a_1) = f(a_2)$ and put $x = a_1$ and $x = a_2$ in the above formula. It is easily shown that $a_1 = a_2$
Note: This trick is very useful in functional equations - when you have a variable x outside the functions and everything else inside the function calls.
If $x \neq 1$,

$$\begin{aligned} P(x, \frac{f(x) - x}{1 - x}) &\implies f(x * \frac{f(x) - x}{1 - x} + f(x)) + f(\frac{f(x) - x}{1 - x}) = xf(\frac{f(x) - x}{1 - x}) + f(x + \frac{f(x) - x}{1 - x}) \\ &\implies f(\frac{f(x) - x^2}{1 - x}) + f(\frac{f(x) - x}{1 - x}) = xf(\frac{f(x) - x}{1 - x}) + f(\frac{f(x) - x^2}{1 - x}) \\ &\implies f(\frac{f(x) - x}{1 - x}) = xf(\frac{f(x) - x}{1 - x}) \implies f(\frac{f(x) - x}{1 - x}) = 0 \end{aligned}$$

We already know, $0 = f(2)$ and so, by injectivity, $\frac{f(x)-x}{1-x} = 2$.
And so $f(x) = 2 - x\ \forall x \neq 1$.
For x = 1, we can show:

$$P(1, x) \implies f(x + f(1)) = f(1 + x)$$

Again, via injectivity, $x + f(1) = x + 1 \implies f(1) = 1 = 2 - x$.
And so $\boxed{S1 : f(x) = 2 - x \quad \forall x}$, which indeed fits.

Case ii: If $f(0) = 0$
If $f(a) = f(b) = c$, subtracting $P(a, b)$ from $P(b, a)$: we get $c(a - b) = 0$ and so $f(a) = f(b) = 0$ or $a = b$. Equation (i).
$P(x, 0) \implies f(f(x)) = f(x)$ and so $f(x) = 0$ or $f(x) = x$ using eqn (i).
If $f(1) = 0$, $P(1, x) \implies f(x) = f(1 + x)$.
And so $\boxed{S2 : f(x) = 0 \quad \forall x}$, which indeed fits.

If $f(1) = 1$: claim that $f(x) = x\ \forall x$.
Suppose not, $\exists k \neq 0, f(k) \neq k \implies f(k) = 0$.

Then $P(1 - k, k) \implies f(k - k^2 + f(1 - k)) = 1 = f(1)$.
$f(k - k^2 + f(1 - k)) = f(1) = 1 \neq 0 \implies k - k^2 + f(1 - k) = 1$, from eqn (i).
If $f(1 - k) = 0$, we get $k - k^2 = 1 \Rightarrow\Leftarrow$.
If $f(1 - k) = 1 - k$, we get $k - k^2 + 1 - k = 1 \Rightarrow\Leftarrow$, since $k \neq 0$.

Therefore, there is no $k \neq 0$ such that $f(k) = 0$.
This means $f(x) = x$ for all $x \neq 0$.
Since $f(0) = 0$, we have $f(x) = x$ for all x.
No such x and $\boxed{S3 : f(x) = x \quad \forall x}$, which indeed fits.

5. **Be real.** Given: $a_0, a_1, \ldots, a_{100}$ are positive real numbers

$$P_k(x) = a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k$$

where the indices are taken modulo 101

Let $n = 50$. For the sake of contradiction, assume that each of these polynomials has all real roots; these roots must be negative. Let

$$-\alpha_{1,k}, -\alpha_{2,k}, \ldots, -\alpha_{2n,k}$$

be the roots of the polynomial

$$a_{2n+k}x^{2n} + 2na_{2n-1+k}x^{2n-1} + a_{2n-2+k}x^{2n-2} + a_{2n-3+k}x^{2n-3} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k$$

Indices are taken modulo $2n+1$, so $a_{2n+k} = a_{k-1}$ and $a_{2n-1+k} = a_{k-2}$. Then

$$\sum_{j=1}^{2n} \alpha_{j,k} = 2n \cdot \left(\frac{a_{k-2}}{a_{k-1}}\right); \quad \prod_{j=1}^{2n} \alpha_{j,k} = \frac{a_k}{a_{k-1}}$$

Since the $\alpha_{j,k}$'s are positive, AM-GM inequality can be applied and by virtue of it we are led to

$$\left(\frac{1}{2n}\sum_{j=1}^{2n} \alpha_{j,k}\right)^{2n} \geq \left(\prod_{j=1}^{2n} \alpha_{j,k}\right)$$

$$\implies \left(\frac{a_{k-2}}{a_{k-1}}\right)^{2n} = \frac{a_k}{a_{k-1}}$$

for each k. As both sides of the inequalities are positive, multiplying them we obtain

$$\prod_{k=0}^{2n} \left(\frac{a_{k-2}}{a_{k-1}}\right)^{2n} \geq \prod_{k=0}^{2n} \frac{a_k}{a_{k-1}}$$

But both sides are equal to 1. Therefore all the $2n+1$ A.M.-G.M inequalities are equalities implying that for each k,

$$\alpha_{1,k} = \alpha_{2,k} = \cdots = \alpha_{2n,k} = \frac{a_{k-2}}{a_{k-1}}$$

Since $n \geq 2$, using Vieta's relations gives

$$\frac{a_{k-3}}{a_{k-1}} = \sum_{1 \leq i < j \leq 2n} \alpha_{i,k}\alpha_{j,k} = \binom{2n}{2}\left(\frac{a_{k-2}}{a_{k-1}}\right)^2$$

Simplifying leads

$$\binom{2n}{2}\left(\frac{a_{k-2}}{a_{k-1}}\right)^2 = \frac{a_{k-3}}{a_{k-1}} \implies \binom{2n}{2}(a_{k-2})^2 = a_{k-1}a_{k-3}$$

for each k. Multiplying all these equations yields

$$\left(\binom{2n}{2}^{2n+1} - 1\right)\left(\prod_{k=0}^{2n} a_k\right)^2 = 0$$

which shows that at least one $a_k = 0$, a contradiction.

6. **Add and Subtract** Evaluating the RHS, we see that it becomes 1 for $x < 3$ and 9 for $x > 3$. By means of integration, we get:

$$|f(x)| + g(x) = \begin{cases} 9x + k_0, & \text{if } x > 3 \\ x + k_1, & \text{if } x < 3 \end{cases}$$

Obersvation i: f(x) and g(x) can have atmost degree 1 each
Proof: Since f(x) and g(x) are polynomials, they must be continuous functions. The only way for it to evaluate to the piecewise function on RHS, is if $f(x) < 0 \ \forall x > 3$ and $f(x) > 0 \ \forall x > 3$, or vice-versa.
For any arbitrary $n \in \mathbb{N}, n \geq 2$, let a_n and b_n be the coefficients of x^n in f and g respectively.
Since, f takes different signs before and after 3, we have $a_n + b_n$ and $-a_n + b_n$ as the coefficients of x^n in $|f(x)| + g(x)$ (before and after 3, or vice-versa)
Since $n \geq 2$, we have $a_n + b_n = 0$ and $-a_n + b_n = 0$, since no x^n terms are on RHS for $n \geq 2$.
Therefore, $a_n = 0, b_n = 0$.

Since f is linear, and $|f(x)|$ is non-differentiable at 3. We get f has a zero at 3, so $f(x) = a(x-3)$ as the only solution.
We take $g(x) = bx + c$ for the other linear polynomial.
WLOG, assume $a > 0$. $-a(x-3)$ will be the only other solution to f(x), since $|y| = |-y|$.

$$|a(x-3)| + (bx+c) = \begin{cases} a(x-3) + (bx+c) = (b+a)x + (c-3a), & \text{if } x > 3 \\ -a(x-3) + bx + c = (b-a)x + (c+3a), & \text{if } x < 3 \end{cases}$$

Comparing coefficients, we get $b + a = 9, b - a = 1$, therefore $b = 5, a = 4$
We also have, $g(0) = b(0) + c = c = 12$.
Therefore, the polynomials are $g(x) = 5x + 12$ and $f(x) = \pm 4(x-3)$.